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Generalized Einstein manifolds

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Abstract

A Finslerian manifold is called a generalized Einstein manifold (GEM) if the Ricci directional curvature $R(u, u)$ is independent of the direction. Let $F^0(M, g_t)$ be a deformation of a compact n-dimensional Finslerian manifold preserving the volume of the unitary fibre bundle $W(M)$. We prove that the critical points $g_0 \in F^0(g_t)$ of the integral $I(g_t)$ on $W(M)$ of the Finslerian scalar curvature (and certain functions of the scalar curvature) define a GEM. We give an estimate of the eigenvalues of Laplacian Δ defined on $W(M)$ operating on the functions coming from the base when (M, g) is of minima fibration with a constant scalar curvature $\tilde{H}$ admitting a conformal infinitesimal deformation (CID). We obtain $\lambda \geq \tilde{H}/(n-1)$ $(\Delta f = \lambda f)$. If M is simply connected and $\lambda = \tilde{H}/(n-1)$, then (M, g) is Riemannian and is isometric to an n-sphere. We first calculate, in the general case, the formula of the second variationals of the integral $I(g_t)$ for $g = g_0$, then for a CID we show that for certain Finslerian manifolds, $I''(g_0) > 0$. Applications to the gravitation and electromagnetism in general relativity are given. We prove that the spaces characterizing Einstein–Maxwell equations are GEMs.

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Contents

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1. Introduction

Much important research has been made to find the right geometrization of the equations of electromagnetism and gravitation in general relativity. In 1915 Hilbert [9] announced the two axioms which permit one to deduce the equations of Einstein with the second member thanks to a variational principle (Misner–Thorn–Wheeler, pp. 431–434) [15]. This method of deformation is applied to different and heterogeneous action functionals defined on the space–time M_4 [6,15]. An analogous procedure of the standard model of Kaluza [10], Klein [11] ($M_4 \times T$), applied to the total scalar curvature of a pseudo-Riemannian metric of dimension five determines the equations of Einstein–Maxwell in the absence of matter and electric charge, *the case of pure electromagnetic field* (Lichnerowicz, pp. 197–198) [13]. The action functional is as in the preceding case, an element of the ring of functions defined on space–time M_4. When the stress-energy tensor T represents the case of *pure matter electromagnetic field* ($T = \rho u \otimes u + \dot{\tau}$), the solutions proposed by [6,13] do not seem satisfactory. That is why we pose the problem in the following form:

To find a geometric space and an action function depending on the curvature of this space, such that the critical point of deformed total action function is a solution of the equations of Einstein–Maxwell.

A solution to this problem is the object of this work. In order to know the difference between our viewpoint, and one usually proposed, we will observe that:

(1) The geometric space in question is the unitary tangent bundle on the space–time M_4.

(2) The action functional is constructed by means of a certain scalar curvature of a connection of directions. It is a priori an element of the ring of functions defined on the unitary tangent bundle $W(M_4)$.

(3) The equations of Maxwell with source ($\delta F = \mu_1 u$) are obtained by means of Bianchi identities relative to the connection of directions via the equations of Einstein.

In order that the spaces which characterizes the equations of Einstein with the second member defined above, are contained in a bigger class of Finsler manifold which we call here the generalized Einstein manifolds (GEMs).

We are now going to give an overview of our work. After a brief recall of Finslerian geometry, we deform the Finslerian metric and calculate the first variational of the volume element of the unitary Finslerian fibre bundle $W(M)$. In the compact case, we prove a lemma which permits us to find the variational of the volume $W(M)$. In Sections 2.3 and 2.4, we calculate the variationals of the Finslerian connection and the corresponding curvature tensors. To the Ricci tensor of the Finslerian connection we associate a function with scalar values $H(u,u)$ on $W(M)$, which we call the *directional Ricci curvature*. This function is the same for the connections of Finsler and Berwald and is homogeneous of degree zero, and plays an important role in what follows (it is the analogue in the Riemannian case of the first member of the Poisson equation in the geometric formulation of the Newtonian gravitation ([15] p.300)). We deduce from it by vertical derivation a symmetric tensor of the second order $\tilde{H}_{jk}$ which plays the same role as the Ricci tensor of the Riemannian geometry. Let (M, g_t) be a deformation of the Finslerian metric and $\lambda(x)$ a differentiable function on M. The lemma of Section 2.5 gives us the variational of $\lambda(x)H(u,u)$. Then we calculate the expression of the trace of variationals of $\tilde{H}_{jk}$ as well as of the Ricci tensor of the Berwald connection H_{jk}. In Section 2.6 we suppose that M is compact, and we denote by $F^0(g_t)$ a deformation of the Finslerian metric which preserves the volume of $W(M)$. By means of the Ricci curvature $\tilde{H}_{jk}$ we define on $W(M)$ the scalar-valued function $\tilde{H}_t = \text{trace}(\tilde{H}_t) - \lambda(x)\tilde{H}_t(u,u)$ depending on t and the corresponding integral $I(g_t)$. We prove that $g_0 \in F^0(g_t)$ is a critical point of $I(g_t)$ if and only if $\tilde{H}_{jk}$ is proportional to the metric tensor $\tilde{H}_{jk} = C(z)g_{jk}$ $(z \in W(M))$ $(t = 0, g = g_0)$. We conclude from it that C is a function independent on the direction. Such a manifold will be called a *GEM* (Theorem 1). In Section 2.7 we study the particular cases when λ is constant. In order to simplify the variational calculus of the Finslerian scalar curvature, we choose a deformation which leaves invariant the torsion tensor and we then calculate the trace of the variational of Finslerian Ricci tensor. We prove, as before, that the critical points of the total Finslerian scalar curvature define again a *GEM* (Theorem 2). In Section 3.1, we determine the conditions in order that the Finslerian manifold be a totally geodesic fibration (and minima) (Theorem 3). In the compact case we give an estimate of the eigenvalues of the Laplacian Δ defined on $W(M)$ operating on the functions coming from the base M. When (M, g) of constant scalar curvature $\tilde{H} = \text{trace}(\tilde{H}_{ij})$, has a minimal fibration and admits a conformal infinitesimal deformation (CID). We prove the inequality $\lambda \geq \tilde{H}/(n-1)$ $(\Delta f = \lambda f)$. Besides, if M is simply connected and there is equality, then (M, g) is isometric to an n-sphere (Theorem 4). This theorem generalizes an analogous result in the Riemannian case [14], the method of proof used here being entirely different. In Section 4.1 we obtain the formula of the second variationals of the integral $I(g_t)$ in the case where (M, g_0) is a GEM and $\lambda = \frac{1}{2}n$. In Section 4.2 we study the case of a CID and prove that for certain generalized Einstein metrics the second variational is positive (Theorem 5). The rest of the work is devoted to the application of the preceding method to the solution of the problem posed at the beginning. Let (M, g) be a compact pseudo-Riemannian. We denote by ω the lifting of the pseudo-Riemannian connection on the tangent bundle. Let F be a skew-symmetric 2-tensor. To the pair (ω, F) we associate a connection of directions, without torsion, denoted π on the unitary bundle $W(M)$, admitting two curvature tensors H and G. We consider then a deformation of the metric g leaving unchanged the 2-tensor F. After having proved a lemma analogous to Lemma 3 of Section 2.5. We take up the variational problem similar to Section 2.6 and characterize in this case the GEM (Theorem 6). In Section 5.4 we proceed to the identification of the elements introduced with the elements coming from gravitation and electromagnetism.

2. Generalized Einstein manifolds

2.0. Preliminaries

Let M be a connected, paracompact, n-dimensional manifold of $\mathbb{C}^\infty$ class. Let $TM \to M$ be the tangent bundle and $p: V(M) \to M$ the tangent bundle of non-zero vectors of TM. Let $p^{-1}TM \to V(M)$ be the fibre bundle induced from TM by p. A point of $V(M)$ will be denoted by $z = (x, v)$ where $x = pz \in M$ and $v \in Tpz(M)$. We denote by $TV(M)$ the tangent bundle to $V(M)$. Let (x^i) $(i = 1, \ldots, n)$ be a local chart of the domain $U \subset M$ and (x^i, v^i) the induced local chart on $p^{-1}(U)$ where: $v = v^i(\delta/\delta x^i) \in Tpz(M)$. We suppose that M is endowed with a Finslerian metric. Such a metric is defined by the data of a function F on TM satisfying the following conditions:

(1) $F > 0$ and $\mathbb{C}^\infty$ on $V(M)$,

(2) $F(x, \lambda v) = \lambda F(x, v), \quad \lambda \in \mathbb{R}^+$,

(3) $g_{ij}(x, v) = \dfrac{1}{2}\dfrac{\partial^2 F^2}{\partial v^i \partial v^j}$

is *positive definite*. It is called *pseudo-Finslerian* if g_{ij} defines a non-degenerate quadratic form $(\det(g_{ij}) \neq 0)$. It is clear that g_{ij} is a homogeneous tensor of degree zero in v and we have

$$g_{ij}(z)v^i v^j = g_z(v, v) = F^2, \tag{2.1}$$

where $g_z(\ ,\ )$ denotes the local scalar product in $z \in V(M)$. Henceforward ∇ will denote the Finslerian connection associated to g [1]. It defines a covariant derivative of the fibre bundle $p^{-1}TM \to V(M)$. We call the indicatrix in $x \in M$, the hypersurface S_x in $T_x(M)$ defined by the equation $F(x, v) = 1$. We denote by $W(M) = U_{x \in M} S_x$ the fibre bundle of unitary tangent vectors to M. Let $u : M \to W(M)$ be the unitary vector fields and $\omega = \Sigma u_i \, \mathrm{d}x^i$ the corresponding 1-form. We denote by $(\mathrm{d}\omega)^{n-1}$ the $(n-1)$th exterior power of $\mathrm{d}\omega$. The volume element of the fibre bundle $W(M)$ will be represented by a $(2n-1)$-form on $W(M)$ [1]:

$$\eta = \frac{(-1)^N}{(n-1)!}\Phi, \quad \Phi = \omega \wedge (\mathrm{d}\omega)^{n-1}, \quad N = \frac{n(n-1)}{2}. \tag{2.2}$$

We suppose M compact, as in the theory of the harmonic forms, we introduce on the differential forms defined on $W(M)$, the codifferential operator δ, adjoint to d, in global scalar product defined on $W(M)$. If $\pi_1 = a_i(z)\, \mathrm{d}x^i, z \in W(M)$, is a horizontal 1-form on $W(M)$, we have proved in [1] that

$$\delta\pi_1 = -(\nabla^j a_j - a_j \nabla_0 T^j), \tag{2.3}$$

where T^j is the vector trace of the torsion tensor. Similarly, $\pi_2 = b_j \nabla u^j$ $(b_j v^j = 0)$ is a vertical 1-form on $W(M)$, we have [2]

$$\delta\pi_2 = -F(\dot\nabla_j b^j + b^j T_j) = -F g^{ji} \dot\delta_j b_i \quad \left(\dot\delta_j = \delta/\delta v^j\right). \tag{2.4}$$

Henceforward, we denote by $(\nabla_k, \dot\nabla_k)$ the components of the Finslerian covariant derivative with respect to coframe $(\mathrm{d}x^k, \nabla v^k)$ and frequently use the formulas (2.3) and (2.4).

2.1. Variationals of the volume element of the unitary fibre bundle

A deformation of a Finslerian metric will mean a one-parameter family of this metric. Supposing the deformation of the metric, ω as well as η depend on the parameter $t \in [-\varepsilon, \varepsilon], \varepsilon$ sufficiently small > 0 we will calculate the derivative of η with respect to t.

First of all we have

$$\omega_t = \frac{\delta F_t}{\delta v^i}\,\mathrm{d}x^i, \quad \omega' = \frac{\delta F'}{\delta v^i}\,\mathrm{d}x^i \qquad (\delta_i^\bullet = \partial_i^\bullet), \tag{2.5}$$

where the notation $'$ denotes the derivative with respect to t. This derivative commutes with the differentiation d, so from (2.2) we have

$$\Phi' = \omega' \wedge (\mathrm{d}\omega)^{n-1} + (n-1)\omega \wedge (\mathrm{d}\omega)^{n-2} \wedge \mathrm{d}\omega'. \tag{2.6}$$

By a simple calculation from (2.5), we obtain

$$\delta F'/\delta v^i = g'_{ir}u^r - (F'/F)u_i \quad (u = F^{-1}v), \tag{2.7}$$

$$\omega' = g'_{ir}u^r\mathrm{d}x^i - (F'/F)\omega. \tag{2.8}$$

Let us denote by $\theta = \nabla v$, and $\beta = \nabla u$, we have

$$\beta^j = F^{-1}(\delta^j_{\ k} - u^j u_k)\theta^k \quad (u_j\beta^j = 0). \tag{2.9}$$

From (2.5) we get by differentiation

$$\mathrm{d}\omega' = \frac{\partial^2 F'}{\delta v^i \partial x^j}\,\mathrm{d}x^j \wedge \mathrm{d}x^i + F\frac{\delta^2 F'}{\delta v^i \delta v^j}\beta^j \wedge \mathrm{d}x^i, \tag{2.10}$$

where $(\partial/\partial x^j, \delta/\delta v^j)$ denotes the pfaffian derivatives with respect to $(\mathrm{d}x^j, \theta^j)$, define at $z \in V(M)$ a basis of $T_z V(M)$. The first term of the right-hand side of (2.10) is a 2-form in $\mathrm{d}x$, by putting it in (2.6), it cancels the second term. The coefficients of the second term are given by

$$F\frac{\delta^2 F'}{\delta v^i \delta v^j} = g'_{ij} - \frac{F'}{F}g_{ij} - g'_{jr}u^r u_i - g'_{ir}u^r u_j + \frac{3F'}{F}u_i u_j. \tag{2.11}$$

Taking account of (2.8), (2.10) and (2.11), the derivative of Φ can be written as

$$\begin{aligned}\Phi' = &-n\left(F'/F\right)\Phi + g'_{ir}u^r\,\mathrm{d}x^i \wedge (\mathrm{d}\omega)^{n-1} \\ &+(n-1)\omega \wedge (\mathrm{d}\omega)^{n-2} \wedge g'_{ij}\beta^j \wedge \mathrm{d}x^i,\end{aligned} \tag{2.12}$$

where

$$F'/F = \tfrac{1}{2}g'_{ij}u^i u^j. \tag{2.13}$$

To evaluate the last two terms of the right-hand side of (2.12) we take an orthonormal frame (e_i) $(i = 1, \ldots, n)$ at $x \in M$ such that, $u = e_n$, we have $u^n = 1, u^\alpha = 0, \beta_n = 0, \beta_\alpha = \omega_{\alpha n}(\alpha = 1, \ldots, n-1)$ where ω_{ij} is the Finslerian connection. Thus the last term of the right-hand side of (2.12) is $g^{i\alpha}g'_{i\alpha}\Phi$ and the last but one term is equal to $g^{in}g'_{in}\Phi$, thus their sum is $g^{ij}g'_{ij}\Phi$. Dividing the sides of (2.12) by $(n-1)!$ we get the following lemma.

Lemma 1. *The first variational of the volume element of the Finslerian unitary fibre bundle is defined by*

$$\eta' = \left(g^{ij} - \tfrac{1}{2}nu^i u^j\right)g'_{ij}\eta. \tag{2.14}$$

2.2. Compact case

Lemma 2. *Let (M, g) be a compact Finslerian manifold and f a differentiable function homogeneous of degree zero in v on $W(M)$ and $t = g'$. Then we have the formula*

$$\int_{W(M)} f \cdot \mathrm{trace}(t) \cdot \eta = \int_{W(M)} \left[nf + \tfrac{1}{2}F^2 g^{ij}\delta_i^\bullet \delta_j^\bullet f\right] t(u,u)\eta. \tag{2.15}$$

Proof. Let f be a differentiable function on W and $t_{ij} = g'_{ij}$. Consider the field of covectors defined by its components

$$Y_j = f \cdot t_{ij}u^i.$$

To Y we associate the vertical 1-form on $W(M)$ defined by

$$\hat{Y} = Y - u \cdot g(Y,u), \qquad \hat{Y}_0 = v^r \hat{Y}_r = 0, \tag{2.16}$$

where 0 denotes the multiplication contracted by v. Thanks to the homogeneity of the torsion tensor we have

$$Fg^{ij}\delta_i^\bullet \hat{Y}_j = g^{ij}\delta_i^\bullet f t_{0j} + f \cdot \mathrm{trace}(t) - nf t_{ij}u^i u^j \quad \left(\delta_i^\bullet = \delta/\delta v^i\right). \tag{2.17}$$

However,

$$g^{ij}\delta_i^\bullet f t_{0j} = \tfrac{1}{2}Fg^{ij}\delta_i^\bullet\left(\delta_j^\bullet f t_{00}F^{-1}\right) - \tfrac{1}{2}g^{ij}t_{00}\delta_j^\bullet\delta_i^\bullet f.$$

Substituting this expression in (2.17) we get

$$\begin{aligned} f \cdot \mathrm{trace}(t) = &\left(nf + \tfrac{1}{2}F^2 g^{ij}\delta_i^\bullet\delta_j^\bullet f\right) t_{00}F^{-2} + Fg^{ij}\delta_i^\bullet \hat{Y}_j \\ &-\tfrac{1}{2}Fg^{ij}\delta_j^\bullet\left(\delta_i^\bullet f t_{00}F^{-1}\right). \end{aligned} \tag{2.18}$$

By (2.4), the last two terms of the right-hand side of this equation are divergences over $W(M)$ [2]. Since M is compact we obtain the lemma by integration over $W(M)$. If $f = \varphi$ is a function on M we have

$$\int_{W(M)} \varphi \cdot \mathrm{trace}(t) \cdot \eta = n \int_{W(M)} \varphi t(u,u)\eta, \tag{2.19}$$

where $t(u,u) = t_{ij}u^i u^j$. Now vol $W = \int_{W(M)} \eta$ and by putting $\varphi = 1$ in (2.19) we get

$$\begin{aligned} (\mathrm{vol}\, W)' &= \int_{W(M)} \left[\mathrm{trace}(t) - \tfrac{1}{2}nt(u,u)\right]\eta \\ &= \frac{1}{2}\int_{W(M)} \mathrm{trace}(t) \cdot \eta = \frac{n}{2}\int_{W(M)} t(u,u)\eta. \qquad \square \end{aligned} \tag{2.20}$$

2.3. Variationals of a Finslerian connection

Let $U(x^i) \subset M$ be a local chart of M and $p^{-1}(U)(x^i, v^i)$ the induced local coordinates on $W(M)$. The one-parameter family of the Finslerian connection is represented by the matrix [1]

$$\omega^i_j \,|_{p^{-1}(U)} = \Gamma^{*i}{}_{jk}(x, v, t)\,\mathrm{d}x^k + T^i{}_{jk}(x, v, t)\nabla v^k, \tag{2.21}$$

where ∇ is the Finslerian covariant derivative associated to the one-parameter family of the Finslerian metric. We will calculate first the derivative with respect to t, the coefficients Γ of the Finslerian connection. Now these coefficients are defined by [1]

$$\Gamma^{*i}{}_{jk} = \tfrac{1}{2} g^{im}\left(\delta_k g_{mj} + \delta_j g_{mk} - \delta_m g_{kj}\right) - \left(T^i{}_{js}\Gamma^{*s}{}_{0k} + T^i{}_{ks}\Gamma^{*s}{}_{0j} - T_{kjs} g^{im}\Gamma^{*s}{}_{0m}\right) \tag{2.22}$$

and

$$T^i{}_{jk} = \tfrac{1}{2} g^{ir}\delta^{\bullet}_k g_{rj} \quad \left(\delta_k = \frac{\delta}{\delta x^k}, \delta^{\bullet}_k = \frac{\delta}{\delta v^k}\right). \tag{2.23}$$

Now from (2.22), by deriving with respect to t, we get

$$\Gamma'^{*i}{}_{jk} = \tfrac{1}{2} g^{im}\left(\nabla_k t_{mj} + \nabla_j t_{mk} - \nabla_m t_{jk}\right) - \left(T^i{}_{js}G'^s{}_k + T^i{}_{ks}G'^s{}_j - T_{kjs}g^{im}G'^s{}_m\right), \tag{2.24}$$

where $t_{ij} = g'_{ij}$ and $G^i{}_j = \Gamma^{*i}{}_{0j}$. We multiply the two sides of (2.24) by v^j, taking note of the homogeneity of Finslerian torsion tensor

$$\Gamma'^{*i}{}_{0k} = G'^{*i}{}_k = \tfrac{1}{2}\left(\nabla_k t^i{}_0 + \nabla_0 t^i{}_k - \nabla^i t_{0k}\right) - 2T^i{}_{ks}G'^s. \tag{2.25}$$

We multiply this relationship by v^k:

$$\Gamma'^{*i}{}_{00} = 2G'^i = \nabla_0 t^i{}_0 - \tfrac{1}{2}\nabla^i t_{00}. \tag{2.26}$$

Substituting (2.25) and (2.26) in (2.24) we get

$$\begin{aligned} \Lambda^i{}_{jk} &= \Gamma'^{*i}{}_{jk} + T^i{}_{jr}\Gamma'^{*r}{}_{0k} \\ &= \tfrac{1}{2}\left(\nabla_k t^i_j + \nabla_j t^i{}_k - \nabla^i t_{kj}\right) - \tfrac{1}{2}T^i{}_{kr}\left(\nabla_j t^r{}_0 + \nabla_0 t^r{}_j - \nabla^r t_{j0}\right) \\ &\quad + \tfrac{1}{2}T_{kjr}\left(\nabla^i t^r{}_0 + \nabla_0 t^{ir} - \nabla^r t^i{}_0\right) + Q^i{}_{jrk}\left(\nabla_0 t^r{}_0 - \tfrac{1}{2}\nabla^r t_{00}\right), \end{aligned} \tag{2.27}$$

where $Q^i{}_{jrk}$ is the third tensor of curvature and of the Finslerian connection defined by

$$Q^i{}_{jrk} = T^i{}_{ks}T^s{}_{jr} - T^i{}_{rs}T^s{}_{jk}. \tag{2.28}$$

If we derive the two sides of (2.21) with respect to t we get

$$\omega'^i{}_j = \Lambda^i{}_{jk}\mathrm{d}x^k + T'^i{}_{jk}\nabla v^k, \tag{2.29}$$

where Λ is defined by (2.27).

2.4. Variationals of Finslerian curvature tensors

The Finslerian curvature is a 2-form on the unitary fibre bundle $W(M)$ defined by

$$\Omega^i{}_j = \mathrm{d}\omega^i{}_j + \omega^i{}_r \wedge \omega^r{}_j. \tag{2.30}$$

The derivative with respect to t, commuting with the differentiation d is given by

$$\Omega'^i{}_j = \mathrm{d}\omega'^i{}_j + \omega'^i{}_r \wedge \omega^r{}_j + \omega^i{}_r \wedge \omega'^r{}_j. \tag{2.31}$$

In virtue of (2.29),we have

$$\mathrm{d}\omega'^i{}_j = \mathrm{d}\Lambda^i{}_{jl} \wedge \mathrm{d}x^l + \mathrm{d}T'^i{}_{jl} \wedge \nabla v^l + T'^i{}_{jr}\mathrm{d}\nabla v^r. \tag{2.32}$$

Now

$$\nabla v^r = \mathrm{d}v^r + v^h\omega^r{}_h. \tag{2.33}$$

From which, by taking into account (2.30)

$$\mathrm{d}\nabla v^r = \nabla v^h \wedge \omega^r{}_h + \Omega^r{}_0 \quad (\Omega^r{}_0 = v^m\Omega^r{}_m). \tag{2.34}$$

Putting these relationships in (2.32) we have

$$\mathrm{d}\omega'^i{}_j = \mathrm{d}\Lambda^i{}_{jl} \wedge \mathrm{d}x^l + \mathrm{d}T'^i{}_{jl} \wedge \nabla v^l + T'^i{}_{jr}\Omega^r{}_0 - T'^i{}_{jr}\omega^r{}_h \wedge \nabla v^h.$$

Thus (2.31) can be written as

$$\Omega'^i{}_j = \nabla\Lambda^i{}_{jk} \wedge \mathrm{d}x^k + \nabla T'^i{}_{jk} \wedge \nabla v^k - \Lambda^i{}_{jr}T^r{}_{kl}\,\mathrm{d}x^k \wedge \nabla v^l + T'^i{}_{jr}\Omega^r{}_0. \tag{2.35}$$

Now we are going to calculate the derivative of the curvature 2-form Ω, as a function of the derivative of the curvature components. This 2-form can be written [1]

$$\Omega^i{}_j = \tfrac{1}{2}R^i{}_{jkl}\,\mathrm{d}x^k \wedge \mathrm{d}x^l + P^i{}_{jkl}\,\mathrm{d}x^k \wedge \nabla v^l + \tfrac{1}{2}Q^i{}_{jkl}\nabla v^k \wedge \nabla v^l, \tag{2.36}$$

where the R, P and Q are the curvature tensors of the Finslerian connection. The derivative with respect to t of the two sides of (2.36) is

$$\begin{aligned}\Omega'^i{}_j = {}& \tfrac{1}{2}\left(R'^i{}_{jkl} + P^i{}_{jkr}\Lambda^r{}_{0l} - P^i{}_{jlr}\Lambda^r{}_{0k}\right)\mathrm{d}x^k \wedge \mathrm{d}x^l \\ & + \left(P'^i{}_{jkl} + Q^i{}_{jrl}\Lambda^r{}_{0k}\right)\mathrm{d}x^k \wedge \nabla v^l + \tfrac{1}{2}Q'^i{}_{jkl}\nabla v^k \wedge \nabla v^l.\end{aligned} \tag{2.37}$$

By identifying the coefficients of the terms in $\mathrm{d}x \wedge \mathrm{d}x, \mathrm{d}x \wedge \nabla v$ and $\nabla v \wedge \nabla v$ of the two sides of (2.35) and (2.37) we get successively:

$$R'^i{}_{jkl} = \nabla_k\Lambda^i{}_{jl} - \nabla_l\Lambda^i{}_{jk} + P^i{}_{jlr}\Lambda^r{}_{0k} - P^i{}_{jkr}\Lambda^r{}_{0l} + T'^i{}_{jr}R^r{}_{0kl}, \tag{2.38}$$

$$P'^i{}_{jkl} = -\nabla^{\bullet}_l\Lambda^i{}_{jk} - \Lambda^i{}_{jr}T^r{}_{kl} + Q^i{}_{jlr}\Lambda^r{}_{0k} + \nabla_k T'^i{}_{jl} - T'^i{}_{jr}\nabla_0 T^r{}_{kl}, \tag{2.39}$$

$$Q'^i{}_{jkl} = \nabla^{\bullet}_k T'^i{}_{jl} - \nabla^{\bullet}_l T'^i{}_{jk}. \tag{2.40}$$

The formulas (2.38)–(2.40) give us the variationals of the curvature tensors of a Finslerian connection. Finally, let $\pi^i{}_j$ be the one-parameter family of the Berwald connection 1-form associated to g, defined on $p^{-1}(U)$ by [3]

$$\pi^i{}_j = G^i{}_{jk}(x, v, t)\,\mathrm{d}x^k. \tag{2.41}$$

Denoting by H and G the two curvature tensors of this connection and applying the above method we find

$$H'^i{}_{jkl} = D_k G'^i{}_{jl} - D_l G'^i{}_{jk} + G^i{}_{jrk} G'^r{}_l - G^i{}_{jrl} G'^r{}_k, \tag{2.42}$$

where D is the covariant derivative in $\pi^i{}_j$ defined by (2.41).

2.5. *Variationals of some Finslerian scalar curvature tensors*

2.5.1.

Let us recall that the curvature tensor R of the Finslerian connection is related to the tensor H of Berwald connection by [3]

$$\begin{aligned} R^i{}_{jkl} &= H^i{}_{jkl} + T^i{}_{jr} R^r{}_{0kl} + \nabla_l \nabla_0 T^i{}_{jk} - \nabla_k \nabla_0 T^i{}_{jl} \\ &\quad + \nabla_0 T^i{}_{lr} \nabla_0 T^r{}_{jk} - \nabla_0 T^i{}_{kr} \nabla_0 T^r{}_{jl}, \\ R_{jk} &= H_{jk} + T^i{}_{jr} R^r{}_{0ik} + \nabla_k \nabla_0 T_j - \nabla_i \nabla_0 T^i{}_{jk} \\ &\quad + \nabla_0 T^i{}_{kr} \nabla_0 T^r{}_{ij} - \nabla_0 T_r \nabla_0 T^r{}_{jk}. \end{aligned} \tag{2.43}$$

Let us denote by $R_{ij} = R^r{}_{irj}$ and $H_{ij} = H^r{}_{irj}$ the corresponding Ricci tensors, we have by (2.43)

$$R_{ij} v^i v^j = H_{ij} v^i v^j = H(v, v). \tag{2.44}$$

Lemma 3. *Let (M, g_t) be a deformation of a Finslerian manifold and let $\lambda(x)$ be a differentiable function on M, we have the formula*

$$\lambda(x) H'(u, u) = \text{ divergence over } W(M) + \Phi t(u, u), \tag{2.45}$$

where:

$$\Phi = \tfrac{1}{2}\left[\nabla_i \gamma^i - \gamma_i \nabla_0 T^i - F^2 g^{ij} \delta_i^\bullet \left(\psi_j / F\right)\right], \tag{2.46}$$

$$\psi_j = -\left(F^{-1} \nabla_0 \gamma_j + F \delta_j^\bullet f\right), \tag{2.47}$$

$$\gamma_i = 2\lambda(x) \nabla_0 T_i - \nabla_i \lambda - T_i \nabla_0 \lambda, \tag{2.48}$$

$$f = -\left(\tfrac{1}{2}\right) F^{-2} \nabla_0 \nabla_0 \lambda. \tag{2.49}$$

Proof. The derivative of $H(v,v)$ with respect to t is obtained from (2.38) and (2.42):

$$R'_{ij}v^iv^j = H'_{ij}v^iv^j = 2\nabla_i G'^i - \nabla_0 G'^i{}_i + 2\nabla_0 T_i G'^i. \tag{2.50}$$

Let $\lambda(x)$ be a differentiable function on M. Let us calculate $\lambda(x)H'(u,u)$. By (2.50), (2.26) and (2.27) we have

$$\begin{aligned}\lambda(x)H'(u,u) &= 2\left[\nabla_i\left(\frac{\lambda G'^i}{F^2}\right) - \lambda\frac{G'^i}{F^2}\nabla_0 T_i\right] - \nabla_0\left(\frac{\lambda G'^i_i}{F^2}\right)\\ &\quad + \frac{G'^i_i\nabla_0\lambda}{F^2} + \frac{2G'^i}{F^2}(2\lambda\nabla_0 T_i - \nabla_i\lambda)\\ &= \delta(\mu v) - \delta\sigma + F^{-2}\nabla_0 t^i{}_0\gamma_i - \tfrac{1}{2}F^{-2}\nabla^i t_{00}\gamma_i + \tfrac{1}{2}F^{-2}\nabla_0 t^i_i\nabla_0\lambda,\end{aligned} \tag{2.51}$$

where δ denotes the codifferential with respect to the volume element η (2.3). μ and σ are defined by

$$\mu = \lambda(x)G'^i{}_i F^{-2}, \qquad \sigma^i = 2\lambda(x)G'^i F^{-2}. \tag{2.52}$$

The first two terms of the right-hand side of (2.51) are divergences over $W(M)$. We are going to calculate the last three terms:

$$\begin{aligned}F^{-2}\nabla_0 t^i{}_0\gamma_i &= \nabla_0\left(t^i{}_0\gamma_i F^{-2}\right) - t^i{}_0\nabla_0\gamma_i F^{-2},\\ \tfrac{1}{2}\nabla_0 t^i{}_i\nabla_0\lambda F^{-2} &= \tfrac{1}{2}\nabla_0\left(t^i{}_i\nabla_0\lambda F^{-2}\right) - \tfrac{1}{2}t^i{}_i\nabla_0\nabla_0\lambda F^{-2},\\ -\tfrac{1}{2}\nabla^i t_{00}\gamma_i F^{-2} &= -\tfrac{1}{2}\nabla^i\left(\left(t_{00}/F^2\right)\gamma_i\right) + \tfrac{1}{2}t_{00}\gamma_i\nabla_0 T^i F^{-2}\\ &\quad + \tfrac{1}{2}t_{00}\left(\nabla_i\gamma^i - \gamma_i\nabla_0 T^i\right)F^{-2}.\end{aligned}$$

Thus (2.51) can be written as

$$\begin{aligned}\lambda(x)H'(u,u) = {} & \text{divergence over } W(M) + \tfrac{1}{2}t(u,u)\left(\nabla_i\gamma^i - \gamma_i\nabla_0 T^i\right)\\ & + f t^i{}_i - t^i{}_0\nabla_0\gamma_i F^{-2}.\end{aligned} \tag{2.53}$$

On the other hand, we have

$$\begin{aligned}t^i{}_i f = g^{ij}\delta^\bullet_j t_{i0}\cdot f &= F g^{ij}\delta^\bullet_j\left(\frac{t_{i0}}{F}f\right) + \frac{t_{00}}{F^2}f - g^{ij}t_{i0}\delta^\bullet_j f\\ &= F g^{ij}\delta^\bullet_j\hat{Y}_i + n\frac{t_{00}}{F^2}f - t^j{}_0\delta^\bullet_j f,\end{aligned} \tag{2.54}$$

where $\hat{Y}$ is defined by (2.16) and f by (2.49). By putting (2.54) in (2.53) we see that the coefficient of $F^{-1}t^i{}_0$ this relationship is the covector ψ defined by (2.47)

$$\begin{aligned}F^{-1}g^{ij}t_{i0}\psi_j &= \tfrac{1}{2}g^{ij}\delta^\bullet_i t_{00}\psi_j F^{-1}\\ &= \tfrac{1}{2}g^{ij}\delta^\bullet_i\left(F^{-2}t_{00}\psi_j\right)F + \tfrac{1}{2}t(u,u)\psi_0 F^{-1} - \tfrac{1}{2}t(v,v)g^{ij}\delta^\bullet_i\left(F^{-1}\psi_j\right).\end{aligned}$$

Let us put

$$Z_j = \tfrac{1}{2}t(u,u)\psi_j, \qquad \hat{Z}_j = Z_j - u_j Z_0 F^{-1}.$$

Then we have

$$F^{-1}g^{ij}t_{i0}\psi_j = \text{ divergence over } W(M) + \tfrac{1}{2}nF^{-1}t(u,u)\psi_0 \\ -\tfrac{1}{2}g^{ij}t(v,v)\delta_i^{\bullet}\left(F^{-1}\psi_j\right). \tag{2.55}$$

By taking into account (2.53)–(2.55), we obtain the lemma. □

In particular: If λ is constant, we have

$$f = 0, \qquad \gamma_j = 2\lambda\nabla_0 T_j, \qquad \psi_j = -2\lambda F^{-1}\nabla_0\nabla_0 T_j.$$

Thus the formula becomes

$$\lambda H'(u,u) = \text{ divergence over } W(M) + \lambda\tau \cdot t(u,u), \tag{2.56}$$

where

$$\tau = (\nabla^i\nabla_0 T_i - \nabla_0 T_i \nabla_0 T^i) + g^{ij}\delta^{\bullet}_j(\nabla_0\nabla_0 T_i). \tag{2.57}$$

τ is up to a sign, the sum of the codifferential of $\nabla_0 T_i$ and of $F^{-1}\nabla_0\nabla_0 T_i$.

2.5.2.

By means of the function $H(v,v) = H_{ij}v^i v^j$ we construct the tensor of second order defined by

$$\tilde{H}_{jk} = \tfrac{1}{2}\delta_j^{\bullet}\delta_k^{\bullet}H(v,v) = \tfrac{1}{2}\left(H_{jk} + H_{kj} + v^r\delta_j^{\bullet}H_{kr}\right) \quad \left(\delta_j^{\bullet} = \delta/\delta v^j\right). \tag{2.58}$$

Lemma 4. *We have the formula*

$$g^{jk}\tilde{H}'_{jk} = n\tau \cdot t(u,u) + \text{ divergence over } W(M).$$

Proof. By derivation we have

$$\delta_k^{\bullet}H'(v,v) = \delta_k^{\bullet}H'(u,u)F^2 + 2v_k H'(u,u).$$

A second derivation gives us

$$\delta_j^{\bullet}\delta_k^{\bullet}H'(v,v) = F\delta_j^{\bullet}\left[F\delta_k^{\bullet}(H'(u,u))\right] + v_j\delta_k^{\bullet}\left[H'(u,u)\right] \\ +2g_{jk}H'(u,u) + 2v_k\delta_j^{\bullet}\left[H'(u,u)\right]. \tag{2.59}$$

Taking into account (2.58) and the homogeneity of the terms introduced with respect to v, by dividing by two, and by multiplying the two sides by g^{jk} we obtain

$$g^{jk}\tilde{H}_{jk} = \tfrac{1}{2}g^{jk}\delta_j^{\bullet}\delta_k^{\bullet}H'(v,v) = nH'(u,u) + \tfrac{1}{2}Fg^{jk}\delta_j^{\bullet}\left[F\delta_k^{\bullet}(H'(u,u))\right]. \tag{2.60}$$

Now the term $nH'(u,u)$ is defined by the formula (2.56) where $\lambda = n$ and the last term of the right-hand side is a divergence (2.4). □

Lemma 5. *We have the formula*

$$g^{jk}H'_{jk} = g^{jk}\tilde{H}'_{jk} + \text{ divergence over } W(M). \tag{2.61}$$

Proof. By deriving the two sides of (2.58) with respect to t we get

$$g^{jk}H'_{jk} = g^{jk}\tilde{H}'_{jk} - \tfrac{1}{2}g^{jk}v^r\delta^\bullet_j H'_{kr}. \tag{2.62}$$

Now the tensor H satisfies the identity of Bianchi [3]:

$$\delta^\bullet_m H^i{}_{jkl} + D_l G^i{}_{jkm} - D_k G^i{}_{jlm} = 0. \tag{2.63}$$

We contract i and k and multiply by v^l, we obtain taking into account the homogeneity of tensor G

$$v^r\delta^\bullet_m H_{jr} = -D_0\left(G^i{}_{jim}\right) = -D_0 G_{jm} \quad \left(G_{jm} = G^i{}_{jim}\right),$$

whence

$$\begin{aligned}\tfrac{1}{2}g^{jm}v^r\delta^\bullet_m H'_{jr} &= -\tfrac{1}{2}g^{jm}D_0G'_{jm} + g^{jm}\delta^\bullet_s G_{jm}G'^s + g^{jm}G_{sm}G'^s{}_j\\ &= -\tfrac{1}{2}g^{jm}D_0\left(G'_{jm}\right) + Fg^{jm}\delta^\bullet_j\left(G_{sm}G'^s/F\right)\\ &= \text{divergence over } W(M).\end{aligned}$$

Substituting this expression in (2.62) we obtain the lemma. □

2.6. Generalized Einstein manifolds

Let M be a compact Finslerian manifold and $\tilde{H}_{jk}$ the symmetric tensor defined by (2.58). Let λ be a differentiable function on M. We consider the scalar function on $W(M)$ defined by

$$\hat{H} = \tilde{H} - \lambda(x)\tilde{H}(u,u) \quad \left(\tilde{H} = g^{jk}\tilde{H}_{jk}\right). \tag{2.64}$$

Let $F(g_t)$ be a 1-parameter family of Finslerian metric. We denote by $F^0(g_t)$ the subfamily of the metric such that for every $t \in [-\varepsilon, \varepsilon]$ the volume of the unitary fibre bundle corresponding to $g_t \in F^0$ is equal to one. We look for $g_t \in F^0(g_t)$ which makes the integral $I(g_t)$ extremum:

$$I(g_t) = \int_{W(M)} \hat{H}_t\eta_t \tag{2.65}$$

with

$$\int_{W(M)} \eta_t = 1. \tag{2.66}$$

We have successively:

$$\left(\hat{H}_t\right)' = \left(g^{jk} - \lambda u^j u^k\right)' \tilde{H}_{jk} + \left(g^{jk} - \lambda u^j u^k\right) \tilde{H}'_{jk}, \tag{2.67}$$

$$\left(u^j\right)' = -\left(F'/F\right) u^j = -\tfrac{1}{2} t(u,u) u^j \quad \left(t_{ij} = g'_{ij}\right). \tag{2.68}$$

Also

$$\left(u^j u^k\right)' \cdot \tilde{H}_{jk} = -H(u,u) t(u,u).$$

It is easy to see that the derivative of g^{jk} with respect to t is $-\,t^{jk}$. Thus the first term of the right-hand side of (2.67) can be written as

$$\left(g^{jk} - \lambda u^j u^k\right)' \tilde{H}_{jk} = -\left[\tilde{H}^{jk} - \lambda H(u,u) u^j u^k\right] t_{jk}. \tag{2.69}$$

After Lemmas 3 and 4 we have

$$\left(g^{jk} - \lambda u^j u^k\right) \tilde{H}'_{jk} = \text{ divergence over } W(M) + (n\tau - \Phi)\, t(u,u), \tag{2.70}$$

where Φ and τ are defined by (2.54) and (2.57). On the other hand, since the derivative of the volume element η_t with respect to t is defined by (2.14) we have

$$\left(\hat{H}_t \eta_t\right)' = \left(\tilde{H}_t\right)' \eta_t + \hat{H}_t \left[\operatorname{trace}(t) - \tfrac{1}{2} n t(u,u)\right]. \tag{2.71}$$

Thus taking into account (2.69)–(2.71) the derivative of $I(g_t)$ can be written as

$$I'(g_t) = -\langle A, t\rangle = -\int_{W(M)} A^{jk} t_{jk} \cdot \eta_t, \tag{2.72}$$

where $\langle\ ,\ \rangle$ denotes the global scalar product and A is defined by

$$A^{jk} = \tilde{H}^{jk} - \lambda H(u,u) u^j u^k - (n\tau - \Phi) u^j u^k - \hat{H}_t \left(g^{jk} - \tfrac{1}{2} n u^j u^k\right), \tag{2.73}$$

where

$$\hat{H}_t = \tilde{H}_t - \lambda H_t(u,u).$$

The hypothesis that the volume of $W(M)$ is constant, yields after (2.20):

$$\langle D, t\rangle = 0, \qquad D^{jk} = g^{jk} - \tfrac{1}{2} n u^j u^k. \tag{2.74}$$

In order that ($t = 0$) $g_0 \in F^0(g_t)$ gives the extremum of $I(g_t)$ it is necessary and sufficient that there is a constant a such that at $t = 0$:

$$\begin{aligned}&\tilde{H}_{jk} - \lambda H(u,u) u_j u_k - (n\tau - \Phi) u_j u_k - \hat{H}\left(g_{jk} - \tfrac{1}{2} n u_j u_k\right)\\ &\quad = a\left(g_{jk} - \tfrac{1}{2} n u_j u_k\right).\end{aligned} \tag{2.75}$$

By multiplying the two sides by u^j and u^k successively we get

$$H(u,u) - \lambda H(u,u) - (n\tau - \Phi) - \hat{H}\left(1 - \tfrac{1}{2}n\right) = a\left(1 - \tfrac{1}{2}n\right). \tag{2.76}$$

Putting in (2.75) the expression $\lambda H(u,u) + (n\tau - \Phi)$ taken from (2.76) we get

$$\tilde{H}_{jk} - \hat{H}\left(g_{jk} - u_j u_k\right) - H(u,u)u_j u_k = a\left(g_{jk} - u_j u_k\right). \tag{2.77}$$

Multiplying the two sides of (2.77) by v^j we have

$$F^2 \tilde{H}_{0k} = v_k H_{00} \quad \left(H_{00} = H_{ij} v^i v^j\right), \tag{2.78}$$

whence by vertical derivation

$$2v_j \tilde{H}_{0k} + F^2 \tilde{H}_{jk} = g_{jk} H_{00} + 2v_k \tilde{H}_{0j}.$$

Using (2.78) we get

$$\tilde{H}_{jk} = H(u,u) g_{jk}. \tag{2.79}$$

Multiplying the two sides by g^{jk} and contracting:

$$\tilde{H} = g^{jk} \tilde{H}_{jk} = nH(u,u). \tag{2.80}$$

Thus (2.79) can be written as

$$\tilde{H}_{jk} = (1/n)\tilde{H} g_{jk}. \tag{2.81}$$

The left-hand side of this relation is defined by (2.58). From it we deduce by vertical derivation

$$\delta^{\bullet}_m \tilde{H}_{jk} = \delta^{\bullet}_j \tilde{H}_{mk} = (1/n)\left(\delta^{\bullet}_m \tilde{H} g_{jk} + 2\tilde{H} T_{jkm}\right).$$

$\tilde{H}_{jk}$ being homogeneous of degree zero in v satisfying (2.58), by multiplying the two sides of the relation by v^i and v^k taking into account the homogeneity of tensor T, we get

$$\delta^{\bullet}_m \tilde{H} = 0.$$

Thus $\tilde{H}$ and after (2.80), $H(u,u)$ does not depend on the direction. Multiplying the two sides of (2.77) by g^{jk} and using (2.80) we get

$$(1 + \lambda - n)H(u,u) = a. \tag{2.82}$$

λ being a function on M, then $H(u,u) = (1/n)\tilde{H}$ is a function on M. Substituting the values of $H(u,u)$ and $\tilde{H}$ in (2.76) we obtain

$$\left(\lambda - \tfrac{1}{2}n\right)a + (1 + \lambda - n)(n\tau - \Phi) = 0. \tag{2.83}$$

We call the expression $H(u,u)$ the *Ricci directional curvature*.

Definition. A Finslerian manifold is called a *generalized Einstein manifold* if the Ricci directional curvature is independent of the direction. That is to say

$$\tilde{H}_{jk} = C(x) g_{jk}(x, v), \tag{2.84}$$

where $C(x)$ is a function on M.

We have shown the following theorem.

Theorem 1. *The Finslerian metric $g_0 \in F^0(g_t)$ at the critical point $[t = 0, g_0 = g(0)]$ of the integral $I(g_t)$ defines a GEM.*

2.7. Particular cases

(1) We suppose that λ is a *non-zero constant*. From (2.82) it follows that $H(u, u)$ is constant. Thus (2.83) is reduced to

$$\left(\lambda - \tfrac{1}{2}n\right) a + (1 + \lambda - n)(n - \lambda)\tau = 0. \tag{2.85}$$

Hence τ is constant. Now after the expression of τ defined by (2.57) τ is a divergence so that M being compact, by integration on $W(M)$ we find that $\tau = 0$ at $t = 0, a \neq 0$. Thus by (2.85) it follows that $\lambda = \frac{1}{2}n$. By (2.82) we then have at $t = 0$:

$$H(u, u) = \frac{a}{\left(1 - \frac{1}{2}n\right)} = C, \qquad \tilde{H} = \frac{na}{\left(1 - \frac{1}{2}n\right)}, \qquad \hat{H} = \frac{n}{2} \frac{a}{\left(1 - \frac{1}{2}n\right)}. \tag{2.86}$$

Corollary 1. *For λ non-zero constant, the Finslerian metric $g_0 \in F^0(g_t)$ at $t = 0$ is a critical point for the integral $I(g_t)$, and defines at this point a manifold with constant Ricci directional curvature and we have at this point $\tau = 0$.*

(2) *Case $\lambda = 0$.* In this case the integral $I(g_t)$ defined by (2.65) is reduced to

$$I_1(g_t) = \int_{W(M)} \tilde{H}_t \eta_t, \tag{2.87}$$

where $g_t \in F^0(g_t)$ and $\tilde{H}_t = g^{ij} \tilde{H}_{ij}$. The derivative of $I_1(g_t)$ is

$$I_1'(g_t) = -\langle \tilde{A}, t \rangle. \tag{2.88}$$

Therefore

$$\tilde{A}_{ij} = \tilde{H}_{ij} - n\tau u_i u_j - \tilde{H} \left(g_{ij} - \tfrac{1}{2} n u_i u_j\right).$$

Following the reasoning of the preceding section we find that $H(u, u) = a/(1 - n)$. On the other hand, M being compact, by (2.83) we have $\tau = 0$. Therefore $a = 0$. Thus at $t = 0$, we have

$$\tilde{H}_{ij} = 0, \qquad \tau = 0.$$

Corollary 2. *The Finslerian metric $g_0 \in F^0(g_t)$ which makes extremum the integral $I_1(g_t)$ is one for which the Ricci directional curvature is zero and we have at this point $\tau = 0$.*

(3) Let us consider an integral of the form

$$I_2(g_t) = \int_{W(M)} H_t(u,u)\eta_t.$$

Let us look for a metric $g_0 \in F^0(g_t)$ such that $I_2(g_0)$ is an extremum. By a reasoning identical to the previous one, we find that g_0 must satisfy

$$H(u,u)u_iu_j - \tau u_iu_j - H(u,u)\left(g_{ij} - \tfrac{1}{2}nu_iu_j\right) = a\left(g_{ij} - \tfrac{1}{2}nu_iu_j\right),$$

where $a =$ constant. From this relation, it follows immediately that for $t = 0$ we have $\tau = 0$ and $\tilde{H}_{ij} = 0$. We also get the same result as before.

(4) Let (M, g) be a Finslerian manifold with constant sectional curvature in Berwald connection [3].

We suppose also that the torsion tensor satisfies the second-order differential equation

$$\nabla_0\nabla_0 f + 4KF^2 f = 0, \quad f = g^{ij}\delta_i^\bullet T_j, \tag{2.89}$$

where K is constant. Then (M, g) is a GEM with $H(u,u) = (n-1)K$ and $\tau = 0$.

Proof. If (M, g) is a Finslerian manifold of constant sectional curvature in Berwald connection we have [3]

$$H^i{}_{jkl} = K\left(\delta^i{}_k g_{jl} - \delta^i{}_l g_{jk}\right) \quad (\delta \text{ Kronecker symbol}), \tag{2.90}$$

where $K =$ constant and

$$\nabla_0\nabla_0 T^i{}_{jk} + KF^2 T^i{}_{jk} = 0, \tag{2.91}$$

where T is Finslerain torsion tensor. By (2.90) it follows that $H_{jl} = (n-1)Kg_{jl}$ and by (2.90) and (2.91) we obtain with a straightforward calculation

$$-2\tau = \nabla_0\nabla_0 f + 4KF^2 f, \quad f = g^{ij}\delta_i^\bullet T_j. \tag{2.92}$$

By (2.89) the right-hand side of (2.92) is zero. Thus (M, g) is a GEM. □

2.8. Variationals of Finslerian total scalar curvature

2.8.1.

Let (M, g_t) be a deformation of a compact Finslerian manifold. Let S_{jk} be the symmetric tensor defined by

$$S_{jk} = \nabla_0 T^i{}_{kr}\nabla_0 T^r{}_{ij} - \nabla_0 T_i \nabla_0 T^i{}_{jk}. \tag{2.93}$$

In view of studying the variationals of the Finslerian scalar curvature we first prove the following lemma.

Lemma 6. *Let (M, g_t) be a deformation of a Finslerian metric such that the torsion tensor be invariant under this deformation we have*

$$g^{jk} S'_{jk} = \psi(T)\, t(u,u) + \text{ divergence over } W(M), \tag{2.94}$$

where $'$ denotes the derivation with respect to $t \in [-\varepsilon, \varepsilon]$ and $\psi(T)$ is a divergence on $W(M)$ defined by

$$\psi(Z) = -\tfrac{1}{4}\left[\delta Z + \delta(F\nabla_0 Z) - 2\nabla_0 Z_0\right] \tag{2.95}$$

with

$$Z_i = X_i - g^{sk}\delta^\bullet_s Y_{ik}, \tag{2.96}$$

$$X_i = 2\left[g^{jk}\delta^\bullet_i T^r{}_{jk}\nabla_0 T_r - 2\nabla_0 T^{jk}{}_r \delta^\bullet_i T^r{}_{jk} + \delta^\bullet_i T_r \nabla_0 T^r\right], \tag{2.97}$$

$$Y_{ik} = 2\nabla_0 T^{jr}{}_i T_{jrk} - 4\nabla_0 T_{jrk} T^{jr}{}_i + T_i \nabla_0 T_k - \nabla_0 T_i T_k + 2\nabla_0 T_r T^r{}_{ik}.$$

Before proving the lemma, we remark that it remains valid without the hypothesis on the invariance of torsion tensor. In this case one must add to $\psi(T)$ another function of T. We have made the hypothesis in order to reduce the calculations.

Proof. By hypothesis $T' = 0$ we have

$$\left(\nabla_0 T^i{}_{jk}\right)' = -2\delta^\bullet_s T^i{}_{jk} G'^s + T^s{}_{jk} G'^i{}_s - T^i{}_{sk} G'^s{}_j - T^i{}_{js} G'^s{}_k, \tag{2.98}$$

whence

$$g^{jk}\left(\nabla_0 T^i{}_{kr}\nabla_0 T^r{}_{ij}\right)' = 2\nabla_0 T^{jk}{}_i\left[-2\delta^\bullet_s T^i{}_{jk} G'^s + T^s{}_{jk} G'^i{}_s - 2T^i{}_{ks} G'^s{}_j\right]. \tag{2.99}$$

By a calculation identical to the preceding, we get

$$\begin{aligned}-g^{jk}\left(\nabla_0 T_i \nabla_0 T^i{}_{jk}\right)' &= 2\left[g^{jk}\delta^\bullet_s T^i{}_{jk}\nabla_0 T_i + \delta^\bullet_s T_i \nabla_0 T^i\right] G'^s \\ &\quad + \left[T_i \nabla_0 T^s - \nabla_0 T_i T^s + 2\nabla_0 T_r T^{rs}{}_i\right] G'^i{}_s.\end{aligned} \tag{2.100}$$

Taking into account (2.99) and (2.100) we have

$$g^{jk} S'_{jk} = X_s G'^s + Y^s{}_i G'^i_s, \tag{2.101}$$

where X and Y are defined by (2.97). Now the last term of the right-hand side can be written as

$$\begin{aligned}Y^s{}_i G'^i{}_s &= F g^{sk}\delta^\bullet_s\left(Y_{ik} G'^i / F\right) - g^{sk}\delta^\bullet_s Y_{ik} G'^i \\ &= -g^{sk}\delta^\bullet_s Y_{ik} G'^i + \text{divergence over } W(M),\end{aligned}$$

where we have put

$$Y^s_i = g^{sk} Y_{ik}.$$

Thus (2.101) can be written as

$$g^{jk}S'_{jk} = Z_i G'^i. \tag{2.102}$$

Now G'^i is defined by (2.27). The remaining calculations are similar to those made in the proof of Lemma 3. □

2.8.2.

The Ricci tensor of the Finslerian connection is defined by (2.43), whence the scalar curvature

$$R_t = H_t + S_t \tag{2.103}$$

with $R_t = g^{jk}R_{jk};\ H_t = g^{jk}H_{jk};\ S_t = g^{jk}S_{jk}$. As in Section 2.6, let us find a Finslerian metric $g \in F^0(g_t)$ which renders extremum the integral $I_3(g_t)$ defined by

$$I_3(g_t) = \int\limits_{W(M)} R_t \eta_t, \qquad \int\limits_{W(M)} \eta_t = 1. \tag{2.104}$$

Now we have

$$R' = -t^{jk}\left[\tfrac{1}{2}\left(H_{jk} + H_{kj}\right) + S_{jk}\right] + g^{jk}H'_{jk} + g^{jk}S'_{jk}. \tag{2.105}$$

After Lemmas 4 and 5 we have

$$g^{jk}H'_{jk} = n\tau t(u,u) + \text{ divergence over } W(M), \tag{2.106}$$

where τ is defined by (2.57). M is supposed compact on deriving with respect to t the integral $I_3(g_t)$ and using the formulas (2.94) and (2.106) we are led to put

$$B_{jk} = \tfrac{1}{2}\left(H_{jk} + H_{kj}\right) + S_{jk} - \psi_l(T)u_j u_k - R\left(g_{jk} - \tfrac{1}{2}nu_j u_k\right), \tag{2.107}$$

where $\psi_l(T)$ is defined by $\psi_l(T) = n\tau + \psi(T)$.

Thus at $t = 0$, for g_0 to render extremum the integral $I_3(g_t)$ it is necessary and sufficient that we have at this point

$$B_{jk} = b\left(g_{jk} - \tfrac{1}{2}nu_j u_k\right), \tag{2.108}$$

where b is a constant. Let us multiply the two sides of (2.108) by v^j and v^k successively we have

$$(H_{00}/F^2) - \psi_l(T) - R\left(1 - \tfrac{1}{2}n\right) = b\left(1 - \tfrac{1}{2}n\right). \tag{2.109}$$

Substituting the expression of $\psi_1(T)$ defined by (2.109) in (2.108) we get

$$\begin{aligned}&\tfrac{1}{2}\left(H_{jk} + H_{kj}\right) + S_{jk} \\ &\quad = \left(H_{00}/F^2\right)u_j u_k + (R+b)h_{jk}, \quad (h_{jk} = g_{jk} - u_j u_k).\end{aligned} \tag{2.110}$$

Let us multiply both sides of (2.110) by v^j, taking into account the homogeneity of the tensor $S_{jk}(S_{0k} = 0 = S_{k0})$ we have

$$F^2 \tilde{H}_{0k} = H_{00} v_k,$$

whence by vertical derivation

$$\tilde{H}_{jk} = \left(H_{00}/F^2\right) g_{jk}. \tag{2.111}$$

After the reasoning made in Section 2.6, H_{00}/F^2 does not depend on the direction. Thus (M, g_0) is a *generalized Einstein manifold*. On multiplying (2.110) by g^{jk} we get

$$\left(H_{00}/F^2\right) + (n-2)R + (n-1)b = 0. \tag{2.112}$$

Thus it follows that the scalar R is independent from the direction. Let us put in (2.109) the expression of H_{00}/F^2 drawn from (2.112) we have

$$\psi_l(T) = -\left(\tfrac{1}{2}(n-2)\right) R - \tfrac{1}{2} nb. \tag{2.113}$$

We also denote by $F^0(g_t)$, the $g(t)$ of g deformation preserving volume of $W(M)$ and torsion tensor.

Theorem 2. *Let (M, g_t) be a deformation of a compact Finslerian manifold. The Finslerian metric $g_0 \in F^0(g_t)$ which renders extremum the Finslerian total scalar curvature $I_3(g_t)$ defines at this point $(t = 0, g_0)$ a GEM.*

Let us remark that if R is independent of x, then R is constant. From (2.113) it follows that $\psi_l(T)$ is constant. Now $\psi_l(T)$ is a divergence on $W(M)$. Thus $\psi_l(T) = 0$. Therefore $R = -(n/(n-2))b$. It is the case in particular if (M, g) is a Landsberg manifold $(\nabla_0 T^i{}_{jk} = 0)$.

3. Eigenvalues of the Laplacian on the unitary fibre bundle

3.1. Finslerian manifolds whose fibres are totally geodesic or minima

The Finslerian connection defines at each point $z \in V(M)$ a decomposition of the tangent space to $V(M)$ at this point. We have $T_z V(M) = H_z \oplus V_z$ where H_z (respectively V_z) is the horizontal space (respectively vertical). At the point $z \in V(M)$, the Pfaffian derivatives $\left(\partial_k = \delta_k - \Gamma^{*r}{}_{0k}\delta^\bullet_r, \delta^\bullet_k\right)$ defines a frame adapted to the decomposition of $T_z V(M)$. Let us put the Riemannian metric on $V(M)$:

$$\mathrm{d}S^2 = g_{ij}(z)\,\mathrm{d}x^i\,\mathrm{d}x^i + g_{ij}(z)\nabla v^i \nabla v^j, \quad z \in V(M). \tag{3.1}$$

Let $EV(M)$ be the principal fibre bundle of linear frames on $V(M)$ with the structure group $GL(2n, R)$. Let $\hat{D}$ be the Riemannian connection associated to (3.1). This connection has no torsion and $\hat{D}G = 0$. Let $(\pi^{\alpha}{}_{\beta})$ $(\alpha, \beta = i, \bar{i} = 1, 2, \ldots, n)$ be the matrix of this connection relative to the adapted frame, we have

$$\hat{D}(\partial\beta) = \pi^{\alpha}{}_{\beta}\partial_{\alpha} \quad \left(\pi^{\alpha}{}_{\beta} = \Gamma^{\alpha}{}_{\beta\lambda}\sigma^{\lambda}\right), \tag{3.2}$$

where $\sigma^{\lambda} = (\mathrm{d}x^{i}, \nabla v^{i})$. $\hat{D}$ being Riemannian we have

$$\begin{aligned}\Gamma^{\gamma}{}_{\alpha\beta} &= \tfrac{1}{2}G^{\gamma\lambda}(\partial_{\alpha}G_{\beta\lambda} + \partial_{\beta}G_{\lambda\alpha} - \partial_{\lambda}G_{\alpha\beta}) \\ &\quad -\tfrac{1}{2}\left\{G^{\gamma\lambda}[\partial_{\beta}, \partial_{\lambda}]_{\alpha} + [\partial_{\alpha}, \partial_{\beta}]^{\gamma} - G^{\gamma\lambda}[\partial_{\lambda}, \partial_{\alpha}]_{\beta}\right\},\end{aligned} \tag{3.3}$$

where the bracket $[\partial_{\alpha}, \partial_{\beta}]$ is defined by:

$$[\partial_i, \partial_j] = -R^{r}{}_{0ij}\delta^{\bullet}_{r}, \tag{3.4}$$

$$[\partial_i, \delta^{\bullet}_{i}] = G^{r}{}_{ij}\delta^{\bullet}_{r}, \tag{3.5}$$

$$[\delta^{\bullet}_{i}, \delta^{\bullet}_{j}] = 0, \tag{3.6}$$

where the $G^{r}{}_{ij}$ are the coefficients of the connection of Berwald associated to g. Calculating the right-hand side of (3.3) and taking note of the bracket expression, we obtain the 1-form of the Riemannian connection $\pi^{\alpha}{}_{\beta}$ with respect to the adapted frames:

$$\begin{aligned}\pi^{i}{}_{j} &= \omega^{i}{}_{j} + \tfrac{1}{2}g^{ih}R_{0khj}\nabla v^{k}, \\ \pi^{\bar{i}}{}_{j} &= -\left(T^{i}{}_{jk} - \tfrac{1}{2}R^{i}{}_{0jk}\right)\mathrm{d}x^{k} - \nabla_0 T^{i}{}_{jk}\nabla v^{k}, \\ \pi^{i}{}_{\bar{j}} &= \left(T^{i}{}_{jk} + \tfrac{1}{2}R_{0j}{}^{i}{}_{k}\right)\mathrm{d}x^{k} + \nabla_0 T^{i}{}_{j}\nabla v^{k}, \quad \pi^{\bar{i}}{}_{\bar{j}} = \omega^{i}{}_{j},\end{aligned} \tag{3.7}$$

where $\omega^{i}{}_{j}$ represents the 1-form of the Finslerian connection associated to g_{ij} and where T and R the tensors of torsion and curvature of ∇. Let $x = x_0$ be a fixed point of M and the fibre manifold $p^{-1}(x_0) = V^n$ a submanifold of $V(M)$ with Riemannian metric induced:

$$\mathrm{d}\sigma^2 \,|_{p^{-1}(x_0)} = g_{ij}(x_0, v)\,\mathrm{d}v^{i}\mathrm{d}v^{j}. \tag{3.8}$$

Let $\dot{X}$ and $\dot{Y}$ be two tangent vectors in $(x_0, v) \in V^n$ to $p^{-1}(x_0)$ we have

$$\hat{D}_{\dot{Y}}\dot{X} = \dot{D}_{\dot{Y}}\dot{X} + A(\dot{Y}, \dot{X}), \tag{3.9}$$

where $\dot{D}$ is the induced connection and A the second fundamental form of the submanifold $p^{-1}(x_0)$. Let us make explicit the right-hand side. If $\dot{X} = (\delta^{\bullet}_{j})$ and $\dot{Y} = (\delta^{\bullet}_{k})$ we have, with respect to the adapted frame

$$\hat{D}_{\delta^{\bullet}_{k}}\delta^{\bullet}_{j} = \pi^{\alpha}{}_{\bar{j}}(\delta^{\bullet}_{k})\partial_{\alpha} = \pi^{\bar{i}}{}_{\bar{j}\bar{k}}\delta^{\bullet}_{i} + \pi^{i}{}_{\bar{j}\bar{k}}\partial_{i}.$$

Taking into account (3.7) we obtain

$$\hat{D}_{\delta^{\bullet}_{k}}\delta^{\bullet}_{j} = T^{i}{}_{jk}(x_0, v)\delta^{\bullet}_{i} + \nabla_0 T^{i}{}_{jk}(x_0, v)\partial_{i}, \tag{3.10}$$

where the vectors ∂_i and $\delta^{\bullet}_{i}$ are orthogonal with respect to the metric (3.1) $G(\delta^{\bullet}_{i}, \partial_i) = 0$. From this formula it follows immediately that for $V^n = p^{-1}(x_0)$ to be a totally geodesic (respectively minima) submanifold, it is necessary and sufficient that $\nabla_0 T^i{}_{jk} = 0$ (respectively necessary $g^{jk}\nabla_0 T^i{}_{jk} = \nabla_0 T^i = 0$). This condition is equivalent to the vanishing of the second tensor of the curvature P of the Finslerian connection.

Theorem 3. *In order that the fibres of $p: V(M) \to M$ be totally geodesic (respectively minima) it is necessary and sufficient (respectively necessary) that the second tensor of the curvature of the Finslerian connection P (respectively $\nabla_0 T_i = 0$) is zero everywhere.*

3.2. Eigenvalues of the Laplacian on the unitary fibre bundle

Let (M, g) be a compact Finslerian manifold and f a differentiable function on M. By abuse of notation, we will denote also by f its inverse image on $W(M)$. By (2.3) the Laplacian of f is defined by

$$\Delta f = -\left[g^{ij}(z)\nabla_i \nabla_j f - \nabla_i f \nabla_0 T^i\right], \quad z \in W(M). \tag{3.11}$$

Our objective is to study the eigenvalue of Δ for some Finslerian manifolds. A vector field X on M defines a CID for the Finslerian metric if

$$L(\hat{X}) g_{ij} = t_{ij} = 2\varphi(x) g_{ij}(z), \quad x \in M, \ z \in W(M), \tag{3.12}$$

where φ is a function on M, $\hat{X}$ is the lifting of X on $V(M)$ and $L(\hat{X})$ denotes the Lie derivative.

Lemma 7. *We have the formula*

$$F^2 \nabla_i^{\bullet} T_j \varphi^i \varphi^j = \tfrac{1}{2}\delta(\hat{Y} - \hat{Z}) + F^2 \nabla_i^{\bullet} \varphi_j \nabla^{\bullet i} \varphi^j + \tfrac{1}{2}(n-2) T_j \varphi^j \varphi_0, \tag{3.13}$$

where $\varphi_i = \partial_i \varphi$, $\nabla_i^{\bullet}$ denotes the vertical covariate derivative and

$$\hat{Y}_i = F\varphi^r \nabla_r^{\bullet} \varphi_i, \qquad \hat{Z}_i = Z_i - u_i(Z, u), \qquad Z_i = F T_j \varphi^j \varphi_i. \tag{3.14}$$

Proof. First we have, using (2.4)

$$\begin{aligned} F^2 \nabla_i^{\bullet} T_j \varphi^i \varphi^j &= F\nabla_i^{\bullet}\left(F T_j \varphi^j \varphi^i\right) - T_j \varphi^j \varphi_0 + F^2 \left(T_j \varphi^j\right)^2 + F^2 T_r T^r{}_{ij} \varphi^i \varphi^j \\ &= -\delta \hat{Z} + (n-2) T_j \varphi^j \varphi_0 + F^2 T_r T^r{}_{ij} \varphi^i \varphi^j. \end{aligned} \tag{3.15}$$

Now by (2.28) the tensor Q_{ij} can be written as

$$Q_{ij} = T^r{}_{is} T^s{}_{jr} - T_r T^r{}_{ij}. \tag{3.16}$$

Hence

$$F^2 T_r T^r{}_{ij} \varphi^i \varphi^j = F^2 \nabla_i^{\bullet} \varphi^r \nabla_r^{\bullet} \varphi^i - F^2 Q_{ij} \varphi^i \varphi^j.$$

In virtue of this relation (3.15) becomes

$$F^2 \nabla_i^{\bullet} T_j \varphi^i \varphi^j = -\delta \hat{Z} + (n-2) T_j \varphi^j \varphi_0 - F^2 Q_{ij} \varphi^i \varphi^j + F^2 \nabla_i^{\bullet} \varphi^r \nabla_r^{\bullet} \varphi^i. \tag{3.17}$$

Now we will calculate the last term in another way:

$$F^2\nabla_i^\bullet\varphi^r\nabla_r^\bullet\varphi^i = F\nabla_i^\bullet\left(F\varphi^r\nabla_r^\bullet\varphi^i\right) + F^2\varphi^r\nabla_r^\bullet\varphi^i T_i \\ -F^2\varphi^r\left[\nabla_r^\bullet\nabla_i^\bullet\varphi^i + \varphi^i Q_{jr} + \nabla_r^\bullet\varphi^i T_i\right], \tag{3.18}$$

but

$$\nabla_i^\bullet\varphi^i = -\varphi^r T_r, \qquad \nabla_r^\bullet\nabla_i^\bullet\varphi^i = -\nabla_r^\bullet T_i\varphi^i - T^i\nabla_r^\bullet\varphi_i.$$

Thus

$$-F^2\varphi^r\nabla_r^\bullet\nabla_i^\bullet\varphi^i - F^2\varphi^r\nabla_r^\bullet\varphi^i T_i = F^2\nabla_r^\bullet T_i\varphi^i\varphi^r.$$

Therfore (3.18) can be written as

$$F^2\nabla_i^\bullet\varphi^r\nabla_r^\bullet\varphi^i = -\delta\hat{Y} + F^2\nabla_i^\bullet T_j\varphi^i\varphi^j - F^2 Q_{ij}\varphi^i\varphi^j. \tag{3.19}$$

By deducing from (3.15) the relation (3.19) and on dividing by 2, we obtain the lemma. □

Lemma 8. *Let (M, g) be a Finslerian manifold with minima fibration, we have*

$$\varphi\nabla_0(T_j\varphi^j) = \text{ divergence over } W(M) + (2/(n+2))\, F^2\nabla_i^\bullet\varphi_j\nabla^{\bullet i}\varphi^j. \tag{3.20}$$

Proof. By hypothesis, we have

$$\nabla_0 T_i = 0. \tag{3.21}$$

Now the torsion tensor is invariant by a conformal infinitesimal transformation, so its trace. By the Lie derivative of (3.21) we get

$$v_i T^j\varphi_j + T_i\varphi_0 + F^2\nabla_i^\bullet T_j\varphi^j = 0. \tag{3.22}$$

Multiplying the two sides of (3.22) by φ^i and using the formula (3.13) of the preceding lemma, we get

$$\tfrac{1}{2}(n+2)T^j\varphi_j\varphi_0 + F^2\nabla_i^\bullet\varphi_j\nabla^{\bullet i}\varphi^j + \tfrac{1}{2}\delta(\hat{Y} - \hat{Z}) = 0, \tag{3.23}$$

where $\hat{Y}$ and $\hat{Z}$ are defined by (3.14) but

$$T^j\varphi_j\varphi_0 = \nabla_0\varphi T_j\varphi^j = \nabla_0\left[\varphi T_j\varphi^i\right] - \varphi\nabla_0\left(T_j\varphi^j\right) \\ = \text{ divergence over } W(M) - \varphi\nabla_0\left(T_j\varphi^j\right).$$

By putting in (3.23) we obtain the lemma. □

We are now in a position to announce the following theorem.

Theorem 4. *Let (M, g) be a compact Finslerian manifold with minima fibration and with constant scalar curvature $\tilde{H} = g^{ij}\tilde{H}_{ij}$ admitting a CID. If λ is the eigenvalue of the Laplacian Δ operating on the functions of the base M $(\Delta f = \lambda f)$ we have*

$$\lambda > \hat{H}/(n-1). \tag{3.24}$$

If M is simply connected and $\lambda = \hat{H}/(n-1)$, then (M, g) is Riemannian and is isometric to a sphere.

Proof. Let X be a CID. From (2.50) and (3.12) it follows

$$L(\hat{X})H_{00} = -(n-2)\nabla_0\varphi_0 - F^2\Psi \tag{3.25}$$

with

$$\Psi = D_i\varphi^i + D_0(T_i\varphi^i), \tag{3.26}$$

where $\varphi_i = \delta_i\varphi$ and D denote the covariant derivative in the Berwald connection. From (3.25) we get by vertical derivation

$$\delta_i^\bullet L(\hat{X})H_{00} = L(\hat{X})\delta_i^\bullet H_{00} = -2(n-2)D_0\varphi_i - 2v_i\Psi - F^2\delta_i^\bullet\Psi.$$

A second derivation gives us

$$L(\hat{X})\tilde{H}_{ij} = -(n-2)D_i\varphi_j - g_{ij}\Psi - v_i\delta_j^\bullet\Psi - v_j\delta_i^\bullet\Psi - \tfrac{1}{2}F^2\delta_i^\bullet\delta_j^\bullet\Psi, \tag{3.27}$$

where $\tilde{H}_{ij}$ is defined by (2.58). Let us put $\tilde{H} = g^{ij}\tilde{H}_{ij}$. By hypothesis $\tilde{H}$ is constant we have in virtue of (3.27)

$$\begin{aligned}0 = L(\hat{X})\tilde{H} &= -2\varphi\tilde{H} + g^{ij}L(\hat{X})\tilde{H}_{ij}\\ &= -2\varphi\tilde{H} - (n-2)g^{ij}D_i\varphi_j - n\Psi - \tfrac{1}{2}F^2g^{ij}\delta_i^\bullet\delta_j^\bullet\Psi.\end{aligned}$$

Multiplying the two sides by φ and using the condition (3.21), we have

$$\tfrac{1}{2}n\varphi\nabla_0(T_i\varphi^i) + (n-1)\varphi\nabla_i\varphi^i + \varphi^2\tilde{H} + \tfrac{1}{4}F^2g^{ij}\delta_i^\bullet\delta_j^\bullet(\Psi\varphi) = 0.$$

Replacing the expression $\varphi\nabla_0(T_i\varphi^i)$ defined by the previous lemma, in virtue of (2.4) we get

$$\begin{aligned}&\left(\Delta\varphi - \frac{\tilde{H}}{(n-1)}\varphi, \varphi\right) - \frac{n}{(n-1)(n+2)}F^2\nabla_i^\bullet\varphi_j\nabla^{\bullet i}\varphi^j\\ &\quad + \text{divergence over } W(M) = 0,\end{aligned}$$

where (,) denotes the local scalar product. M being compact by integrating on $W(M)$ we get

$$\frac{n}{(n-1)(n+2)}\langle F\nabla_i^\bullet\varphi_j, F\nabla^{\bullet i}\varphi^j\rangle = \langle\Delta\varphi - \frac{\tilde{H}}{n-1}\varphi, \varphi\rangle, \tag{3.28}$$

where $\langle\ ,\ \rangle$ is the global scalar product on $W(M)$. If λ is the eigenvalue of Δ the eigenfunction φ ($\Delta\varphi = \lambda\varphi$). From (3.28) it follows that $\lambda > \tilde{H}/(n-1)$. Let us suppose $\lambda = \tilde{H}/(n-1)$ then

$$\nabla_i^\bullet\varphi_j = 0.$$

From (3.22) we then have $T_i\varphi_0 = 0$. If X is not an isometry, $\varphi_0 \neq 0$ and $T_i = 0$. After a result of Deicke [5] (M, g) is Riemannian. By a theorem of Obata [17] from the fact that M is simply connected, we conclude that (M, g) is isometric to an n-sphere. □

4. Second variationals of the integral $I(g_t)$

4.1. Second variationals of the integral $I(g_t)$

4.1.1.

In order to simplify the calculations we treat the case $\lambda = \frac{1}{2}n$, as indicated in Section 2.7. The function $\hat{H}_t$ under the integral (2.64) reduces to

$$\hat{H}_t = \tilde{H}_t - \tfrac{1}{2}n\tilde{H}_t(u,u), \quad \tilde{H}_t = g^{ij}\tilde{H}_{ij}. \tag{4.1}$$

We put

$$I(g_t) = \int_{W(M)} \hat{H}_t\eta_t, \qquad \int_{W(M)} \eta_t = 1. \tag{4.2}$$

By (2.72) we have

$$I'(g_t) = -\langle A, t\rangle, \tag{4.3}$$

where $t_{jk} = g'_{jk}$ and

$$A_{jk} = \tilde{H}_{jk} - \tfrac{1}{2}nH(u,u)u_ju_k - \tfrac{1}{2}n\tau u_ju_k - \hat{H}\left(g_{jk} - \tfrac{1}{2}nu_ju_k\right). \tag{4.4}$$

For $t = 0$, $I'(g_0) = 0$, (M, g_0) is a GEM $\tilde{H}_{jk} = Cg_{jk}$,

$$C = \frac{a}{1 - \frac{1}{2}n}$$

and we have

$$A_{jk} = a\left(g_{jk} - \tfrac{1}{2}nu_ju_k\right), \tag{4.5}$$

$$\tau\mid_{t=0} = 0. \tag{4.6}$$

At, $t = 0$, $H(u,u)$, $\tilde{H}$ and $\hat{H}$ are defined by (2.86). Before calculating the second derivative of $I(g_t)$ at $t = 0$, we are going to prove some lemmas.

Lemma 9. *Let T' be the deformed torsion tensor of the Finslerian connection and $t = g'$. The following conditions are equivalent:*

(1) $\quad T'^{i}{}_{jk} = 0,$

(2) $\quad \delta^{\bullet}_{k}t^{i}{}_{j} = 0.$

The proof is obtained by a straightforward calculation.

Lemma 10. *Let (M, g_t) be a deformation of a Finslerian metric such that at $t = 0$, the derivative of the torsion tensor is zero. Then we have at this point*

$$t^{jk}\tilde{H}'_{jk}\ |_{t=0} = \tfrac{1}{2}n\,[(n+2)t(u,u) - \mathrm{trace}(t)]\,H'(u,u)\ |_{t=0}$$
$$+\text{divergence over } W(M), \tag{4.7}$$

where

$$H'(u,u) = H'_{ij}u^i u^j.$$

Proof. By formula (2.59) we get

$$\tilde{H}'_{jk} = g_{jk}H'(u,u) + v_k\delta^\bullet_j H'(u,u) + \tfrac{1}{2}v_j\delta^\bullet_k H'(u,u) + \tfrac{1}{2}F\delta^\bullet_j\left[F\delta^\bullet_k H'(u,u)\right]. \tag{4.8}$$

Multiplying the two sides by t^{jk} we get

$$t^{jk}\tilde{H}'_{jk}\ |_{t=0} = \Big\{t^r{}_r H'(u,u) + \tfrac{3}{2}g^{jr}t_{r0}\delta^\bullet_j H'(u,u)$$
$$+\tfrac{1}{2}Ft^{jk}\delta^\bullet_j\left[F\delta^\bullet_k H'(u,u)\right]\Big\}\Big|_{t=0}. \tag{4.9}$$

Now

$$g^{jr}t_{r0}\delta^\bullet_j H'(u,u) = Fg^{jr}\delta^\bullet_j\hat{Z}_r + [nt(u,u) - \mathrm{trace}(t)]\,H'(u,u) \tag{4.10}$$

with

$$Z_r = t_{ri}u^i H'(u,u), \qquad \hat{Z}_r = Z_r - u_r g(Z,u).$$

To calculate the last term of the right-hand side of (4.9) we put

$$Y_r = Ft^k{}_r\delta^\bullet_k H'(u,u), \qquad \hat{Y}_r = Y_r - u_r g(Y,u).$$

Using the hypothesis made in the lemma, we have at $t = 0$:

$$\tfrac{1}{2}Ft^{jk}\delta^\bullet_j\left[F\delta^\bullet_k H'(u,u)\right] = \tfrac{1}{2}Fg^{jr}\delta^\bullet_j\hat{Y}_r + \tfrac{1}{2}(n-1)g(Y,u). \tag{4.11}$$

Putting the formulas (4.11) and (4.10) in (4.9) we obtain the lemma. □

Lemma 11. *Let (M,g) be a Finslerian manifold which satisfies the hypothesis of Lemma* 10. *We have at $t = 0$:*

$$\left[\mathrm{trace}(t) - \tfrac{1}{2}nt(u,u)\right]g^{jk}\tilde{H}'_{jk}\ |_{t=0} = \text{divergence over } W(M)$$
$$+\tfrac{1}{2}n\ \mathrm{trace}(t)H'(u,u)\ |_{t=0}\,. \tag{4.12}$$

Proof. Let us multiply the two sides of (4.8) by g^{jk}, we have

$$g^{jk}\tilde{H}'_{jk} = nH'(u,u) + \tfrac{1}{2}Fg^{jk}\delta^\bullet_j\left[F\delta^\bullet_k H'(u,u)\right]. \tag{4.13}$$

Now at $t = 0$, trace(t) is independent of direction. At this point we have

$$\mathrm{trace}(t)g^{jk}\tilde{H}'_{jk}\ |_{t=0} = n\cdot\mathrm{trace}(t)H'(u,u)\ |_{t=0} + \text{divergence over } W(M). \tag{4.14}$$

Similarly multiplying the two sides of (4.13) by $t(u,u)$ and using the relation (4.10), we find

$$t(u,u)g^{jk}\tilde{H}'_{jk}\mid_{t=0} = \text{trace}(t)H'(u,u)\mid_{t=0} + \text{divergence over } W(M). \tag{4.15}$$

Using formulas (4.14) and (4.15) we get (4.12). □

4.1.2

Now we are going to calculate the second derivative of $I(g_t)$. For this it is clear that the derivative of g^{ij} is $-t^{ij}$. Using the derivative of η defined by (2.14) we obtain

$$\left(A^{ij}g'_{ij}\eta\right)' = \left[-2t^{ir}t^{j}_{r}A_{ij} + A^{ij}g''_{ij} + t^{ij}A'_{ij} + (A,t)\left(g^{ij} - \tfrac{1}{2}nu^iu^j\right)t_{ij}\right]\eta. \tag{4.16}$$

We must evaluate the right-hand side at $t=0$, we have

$$-2t^{ir}t^{j}_{r}A_{ij}\Big|_{t=0} = -2a\left(t^{ij}t_{ij} - \tfrac{1}{2}nt^{i}_{0}t_{i0}F^{-2}\right)\Big|_{t=0}, \tag{4.17}$$

$$(A,t)\left(g^{ij} - \tfrac{1}{2}nu^iu^j\right)t_{ij}\Big|_{t=0} = a\left(\text{trace}(t) - \tfrac{1}{2}nt(u,u)\right)^2\Big|_{t=0}, \tag{4.18}$$

$$A^{ij}g''_{ij}\Big|_{t=0} = a\left(g^{ij}g''_{ij} - \tfrac{1}{2}ng''_{ij}u^iu^j\right)\Big|_{t=0}. \tag{4.19}$$

Now vol $W_t(M) = 1$, by (2.20), $t = g'$ is globally orthogonal at g, so that we have for every $t \in [-\varepsilon, \varepsilon]$:

$$\int_{W(M)} g^{ij}g'_{ij}\eta_t = 0, \tag{4.20}$$

whence it follows, by deriving with respect to t:

$$\int_{W(M)} g^{ij}g''_{ij}\eta = \int_{W(M)} \left\{(t,t) - \text{trace}(t)\left[\text{trace}(t) - \tfrac{1}{2}nt(u,u)\right]\right\}\eta. \tag{4.21}$$

Similarly by (2.20), t is globally orthogonal to the decomposable tensor u^iu^j for every $t \in [-\varepsilon, \varepsilon]$:

$$\int_{W(M)} u^iu^jg'_{ij}\eta_t = 0. \tag{4.22}$$

On deriving this relation, and taking into account (2.68), we have

$$\int_{W(M)} \left\{t(u,u)^2 - u^iu^jg''_{ij} - t(u,u)\left[\text{trace}(t) - \tfrac{1}{2}nt(u,u)\right]\right\}\eta = 0. \tag{4.23}$$

In virtue of (4.21) and (4.23), the relation (4.19) can be written as

$$\langle A, g'' \rangle_{t=0} = a \Big\{ \langle t, t \rangle - \tfrac{1}{2} n \langle t(u,u), t(u,u) \rangle \\ - \langle \mathrm{trace}(t) - \tfrac{1}{2} n t(u,u) \rangle, \mathrm{trace}(t) - \tfrac{1}{2} n t(u,u) \rangle \Big\}_{t=0}. \tag{4.24}$$

It remains to evaluate the term $t^{ij} A'_{ij}$ at $t = 0$. First of all we have

$$(\hat{H})'_{t=0} = \Big\{ \Big(g^{jk} - \tfrac{1}{2} n u^j u^k \Big) \tilde{H}'_{jk} - C \Big[\mathrm{trace}(t) - \tfrac{1}{2} n t(u,u) \Big] \Big\}_{t=0}. \tag{4.25}$$

On the other hand, the relation (2.7) can be written as

$$(u_j)' = t_{jk} u^k - \tfrac{1}{2} t(u,u) u_j. \tag{4.26}$$

Thus $t^{jk} A'_{jk} \mid_{t=0}$ is written as

$$t^{jk} A'_{jk} \mid_{t=0} = \Big\{ t^{jk} \tilde{H}'_{jk} - \Big[\mathrm{trace}(t) - \tfrac{1}{2} n t(u,u) \Big] g^{jk} \tilde{H}'_{jk} \\ + \tfrac{1}{2} n \Big[\mathrm{trace}(t) - \Big(\tfrac{1}{2} n + 1 \Big) t(u,u) \Big] H'(u,u) \\ + C \Big[\Big(\mathrm{trace}(t) - \tfrac{1}{2} n t(u,u) \Big)^2 + n t(u,u)^2 - \tfrac{1}{2} n (t,t) - \tfrac{1}{4} n^2 t(u,u)^2 \\ + n \Big(\tfrac{1}{2} n - 1 \Big) t^j{}_0 t_{j0} F^{-2} \Big] \Big\}_{t=0} - \tfrac{1}{2} n t(u,u) \tau' \mid_{t=0}. \tag{4.27}$$

Taking into account (4.16)–(4.18), (4.24), (4.27), and the Lemmas 10 and 11, the second derivative of $I(g_t)$ at $t = 0$ is obtained:

$$I''(g_0) = \int_{W(M)} \Big[\Phi + \tfrac{1}{2} n t(u,u) \tau' \Big]_{t=0} \eta \\ + \frac{\tilde{H}}{n} \Big[\|t\|^2 - \tfrac{1}{2} n \|t(u,u)\|^2 - \|\mathrm{trace}(t) - \tfrac{1}{2} n t(u,u)\|^2 \Big], \tag{4.28}$$

where $\| \ \|^2 = \langle \ , \ \rangle$ and

$$\Phi = F^{-2} (2\Psi \nabla_0 T_i - \nabla_i \Psi) \Big(\nabla_0 t^i{}_0 - \tfrac{1}{2} \nabla^i t_{00} \Big) \\ + F^{-2} \nabla_0 \Psi \Big[\tfrac{1}{2} \nabla_0 t^r{}_r - T_i \Big(\nabla_0 t^i{}_0 - \tfrac{1}{2} \nabla^i t_{00} \Big) \Big], \tag{4.29}$$

$$\Psi = \tfrac{1}{2} n \Big[\mathrm{trace}(t) - \Big(\tfrac{1}{2} n + 1 \Big) t(u,u) \Big], \tag{4.30}$$

$$\tau = \nabla^i \nabla_0 T_i - \nabla_0 T_i \nabla_0 T^i + g^{ij} \delta^{\bullet}_i (\nabla_0 \nabla_0 T_j).$$

Formula of the second variational. Let (M, g_t) be a deformation of a compact Finslerian manifold. The second derivative of $I(g_t)$ (4.2), for a *GEM* is defined by the formula (4.28).

4.2. The conformal infinitesimal deformation case

In order to study the sign of $I''(g_0)$, we will establish the following lemmas which will permit us to simplify the expression of $I''(g_0)$.

Lemma 12. *Let (M, g_t) be a deformation of a Finslerian metric such that we have at the point $t = 0$:*

$$\nabla_0 T_i \mid_{t=0} = 0, \qquad (\nabla_0 T_i)' \mid_{t=0} = 0. \tag{4.31}$$

Then we have

$$\tau \mid_{t=0} = 0 \quad \textit{and} \quad t(u,u)\tau' \mid_{t=0} = \text{ divergence over } W(M). \tag{4.32}$$

Proof. It is clear that if (M, g) has a minima fibration ($\nabla_0 T_i \mid_{t=0} = 0$) then $\tau \mid_{t=0} = 0$ and $\tau' \mid_{t=0}$ reduced to

$$\tau' \mid_{t=0} = g^{ij}\left[\nabla_i(\nabla_0 T_j)' + \delta_i^\bullet \nabla_0(\nabla_0 T_j)'\right].$$

Hence

$$\begin{aligned} t(u,u)\tau' \mid_{t=0} &= \text{divergence over } W(M) - \nabla^j t(u,u)(\nabla_0 T_j)' \mid_{t=0} \\ &\quad + g^{ij}\delta_i^\bullet \left[t(u,u)\nabla_0(\nabla_0 T_j)'\right] \mid_{t=0} - g^{ij}\delta_i^\bullet t(u,u)\nabla_0(\nabla_0 T_j)' \mid_{t=0} \\ &= \text{divergence over } W(M) \\ &\quad + g^{ij}\left\{\nabla_0\left[\delta_i^\bullet t(u,u)\right] - \nabla_i\left[t(u,u)\right]\right\}(\nabla_0 T_j)' \mid_{t=0} \\ &= \text{divergence over } W(M). \end{aligned}$$

□

Lemma 13. *(M, g_t) be a Finslerian manifold and φ be a differentiable function on M. Then we have the following formulas:*

$$(\mathrm{d}\varphi, \mathrm{d}\varphi) = n(\mathrm{d}\varphi, u)^2 + \text{ divergence over } W(M), \tag{4.33}$$

$$T^i \nabla_i \varphi \nabla_0 \varphi = 1/(n-2)\left[2F^2 Q_{ij}\nabla^i\varphi\nabla^j\varphi + \delta(\hat{Y} + \hat{Z})\right], \tag{4.34}$$

where $\hat{Y}$ and $\hat{Z}$ are defined by (3.14).

Proof. For the first formula we have

$$\begin{aligned} (\mathrm{d}\varphi, \mathrm{d}\varphi) &= \nabla^i\varphi\nabla_i\varphi = F g^{ij}\delta_j^\bullet\left(\nabla_0\varphi\nabla_i\varphi/F\right) + (\mathrm{d}\varphi, u)^2 \\ &= F g^{ij}\delta_j^\bullet \hat{\Psi}_i + n(\mathrm{d}\varphi, u)^2 \\ &= n(\mathrm{d}\varphi, u)^2 + \text{ divergence over } W(M), \end{aligned}$$

where

$$\hat{\Psi}_i = \Psi_i - u_i(\mathrm{d}\varphi, u), \qquad \Psi_i = \nabla_i\varphi(\mathrm{d}\varphi, u).$$

For the relation (4.34) it suffices to add the formula (3.17) to (3.19). Then we obtain (4.34).□

Lemma 14. *Let (M, g_t) be a conformal infinitesimal deformation (CID) of a compact Finslerian manifold. For a generalized Einstein metric g_0 satisfying the conditions of*

Lemma 12 (4.32), *the second variational of $I(g_t)$ at the point g_0 is defined by:*

$$I''(g_0) = (n-1)(n-2) \int_{W(M)} \left[\left(\Delta\varphi - \frac{\tilde{H}}{(n-1)}\varphi, \varphi \right) + \frac{n}{(n-1)(n-2)} F^2 Q_{ij} \nabla^i \varphi \nabla^j \varphi \right] \eta. \tag{4.35}$$

Proof. For a CID we have

$$t_{ij} = 2\varphi g_{ij}, \qquad t(u,u) = 2\varphi, \qquad \mathrm{trace(t)} = 2n\varphi,$$

$$t^i_0 = 2\varphi v^i, \qquad \psi = \tfrac{1}{2} n(n-2)\varphi,$$

$$\Phi = \tfrac{1}{2} n(n-2) \left[(n-2)F^{-2} \nabla_0 \varphi \nabla_0 \varphi + \nabla^i \varphi \nabla_i \varphi + T^i \nabla_i \varphi \nabla_0 \varphi \right].$$

By substituting these expressions in (4.28) and using the preceding lemmas we get the formula (4.35). □

Theorem 5. *On a certain compact Finslerian manifold there exists a generalized Einstein metric for which the second variational of $I(g_t)$ is positive.*

Proof. In the following we suppose that the conditions (4.31) of Lemma 12 are satisfied, and in addition the indicatrix of (M, g_0) is of Einstein type:

$$F^2 Q_{ij} = \frac{F^2 Q}{(n-1)} h_{ij} \quad (h_{ij} = g_{ij} - u_i u_j). \tag{4.36}$$

Then $F^2 Q$ for $n \neq 3$, is independent of v. In fact the third curvature tensor of the Finslerian connection satisfies the Bianchi identy [1]

$$\sigma_{(m,k,l)} \nabla^\bullet_m Q^i{}_{jkl} = 0, \tag{4.37}$$

where σ denotes the sum of the terms obtained by permuting cyclically the indices m, k and l. From it we deduce by contracting i with k on the one hand, and by multiplying the relation so obtained by g^{jm} on the other hand, the formula:

$$\nabla^\bullet_j (F^2 Q^j{}_l) - \tfrac{1}{2} \nabla^\bullet_l (F^2 Q) + v_l Q = 0. \tag{4.38}$$

By putting in (4.38) the expression of Q_{ij} defined by (4.36), we get

$$(n-3) \nabla^\bullet_l (F^2 Q) = 0$$

Thus for $n \neq 3$, $F^2 Q$ is independent of the direction defined by v. Using the formula (4.33) of Lemma 13, we find

$$h_{ij} \nabla^i \varphi \nabla^j \varphi = (1 - 1/n) \nabla_i \varphi \nabla^i \varphi + \text{vertical divergence over } W(M). \tag{4.39}$$

$F^2 Q$ is independent of v, we have

$$\frac{n}{(n-1)} F^2 Q h_{ij} \nabla^i \varphi \nabla^j \varphi = F^2 Q \nabla_i \varphi \nabla^i \varphi + \text{vertical divergence over } W(M). \quad (4.40)$$

Now, we suppose that the second scalar curvature $P = g^{ij} P_{ij}$ of the Finslerian connection is zero. Then we have

$$P = -\tfrac{1}{2} \nabla_0 Q = 0.$$

Thus $F^2 Q$ is constant. Now we know that $F^2 Q < 0$ [2]. For if $F^2 Q \geq 0$, then the CID is trivial: in fact for a Finslerian manifold of minima fibration, we get by (3.20) and (4.34)

$$F^2 Q_{ij} \nabla^i \varphi \nabla^j \varphi + \frac{(n-2)}{(n+2)} F^2 \nabla^{\bullet}_i \varphi_j \nabla^{\bullet i} \varphi^j = \text{divergence over } W(M).$$

M being compact, by integration on $W(M)$ we conclude that $F^2 Q_{ij} \nabla^i \varphi \nabla^j \varphi < 0$. This being so, we choose the torsion tensor in such a way that

$$\frac{F^2 Q}{(n-1)(n-2)} = -\varepsilon.$$

ε sufficiently small > 0. Let $\mu(x)$ be the eigenvalue of Δ for the eigenfunction φ ($\Delta\varphi = \mu\varphi$). By Theorem 4, $\mu(x) > \tilde{H}/(n-1)$. Let μ_1 be the minimum value of μ on M, M being connected and compact $\mu_1 > \tilde{H}/(n-1)$. Taking into account (4.35) and putting $\varepsilon\mu = \varepsilon_1$ we have $\mu - \varepsilon_1 > \tilde{H}/(n-1)$. Thus $I''(g_0) > 0$. □

5. Einstein–Maxwell equations

5.1. A connection of directions associated to the pair (ω, F)

Let (M, g) be an n-dimensional compact pseudo-Reimannian manifold. To the pseudo-Riemannian connection $\omega(\nabla)$ is associated the lift of ω on $V(M)$. It defines a covariant derivation, denoted $\hat{\nabla}$, in the vector bundle $p^{-1}T(M) \to V(M)$. We denote also by $W(M)$ the unitary tangent bundle on M. Let $U(x^i)$ $(i = 1, \ldots, n)$ be a local map of M and F_{ij} be a skew-symmetric tensor on M. To the pair (ω, F) is associated a connection of directions, noted π, locally represented on $p^{-1}(U)$ with local coordinates (x^i, v^i) by

$$\pi^i{}_j = \omega^i{}_j + \xi^i{}_{jk} \, \mathrm{d}x^k = G^i{}_{jk}(x, v) \, \mathrm{d}x^k, \quad (5.1)$$

where

$$\xi^i{}_{jk} = \tfrac{1}{2} K \left(h_{jk} u^r F_r{}^i + u_k F_j{}^i + u_j F_k{}^i \right), \quad (5.2)$$

$$\omega^i_j = \gamma^i_{jk} \, \mathrm{d}x^k, \qquad h_{jk} = g_{jk} - u_j u_k, \qquad \|u\| = 1. \quad (5.3)$$

The (γ^i_{jk}) are the coefficients of the pseudo-Riemannian connection associated to the metric tensor g_{ij} and K is a constant. Let us put

$$\xi^i{}_k = \xi^i{}_{0k} = v^r \xi^i{}_{rk} = \tfrac{1}{2} K \left(F F_k{}^i + u_k F_0{}^i \right), \quad (5.4)$$

$$\xi^i{}_{00} = \xi^i{}_{jk} v^j v^k = KFF_j{}^i v^j = KFF_0{}^i = 2\xi^i, \tag{5.5}$$

whence by vertical derivation

$$\delta_k^\bullet \xi^i = \xi^i{}_{0k}, \qquad \delta_j^\bullet \xi^i{}_k = \xi^i{}_{jk}, \qquad \xi^i{}_{ji} = 0, \tag{5.6}$$

where $F^2 = g_{ij} v^i v^j$.

We will denote also by D the covariant derivation associated to π. D is without torsion and admits, with respect to coframe (dx, Dv) two curvature tensors, which will be denoted, by abuse of notation, by H and G and are written as

$$\begin{aligned} H^i{}_{jkl} &= \left(\delta_k G^i{}_{jl} - G^i{}_{jls} G^s{}_k\right) - \left(\delta_l G^i{}_{jk} - G^i{}_{jks} G^s{}_l\right) \\ &\quad + G^i{}_{rk} G^r{}_{jl} - G^i{}_{rl} G^r{}_{jk} \quad \left(G^i{}_j = v^r G^i{}_{rj}\right), \\ G^i{}_{jkl} &= \tfrac{1}{2} K F^{-1} \left[h_{jk} h^r{}_l + h_{jl} h^r{}_k + h_{kl} h^r{}_j\right] F_r{}^i, \\ H_{jl} &= \left(\delta_i G^i{}_{jl} - G^i{}_{jls} G^s{}_l\right) - \delta_l G^i{}_{ji} + G^i{}_{ri} G^r{}_{jl} - G^i{}_{rl} G^r{}_{ji}, \end{aligned} \tag{5.7}$$

and we have

$$G^i{}_{jil} = 0. \tag{5.8}$$

From the identity (2.63) relative to the connection of directions D [3], it follows in virtue of (5.8), by contracting i and k, and by multiplying by g^{jl} the relation thus obtained:

$$\delta_m^\bullet H = D_i \left(g^{jk} G^i{}_{jmk}\right) - D_i g^{jk} G^i{}_{jmk}. \tag{5.9}$$

Now $\nabla g = 0$, we have

$$D_i g^{jk} = \tfrac{1}{2} K \left(h^j{}_i u^r F_r{}^k + h^k{}_i u^r F_r{}^j + u^k F_i{}^j + u^j F_i{}^k\right). \tag{5.10}$$

Taking into account (5.7) and (5.10), the last term of the right-hand side of (5.9) is zero. Let us put

$$g^{jk} G^i{}_{jmk} = \tfrac{1}{2}(n+1) K Y^i{}_m, \quad Y^i{}_m = F^{-1} h^r{}_m F_r{}^i. \tag{5.11}$$

Now

$$D_i Y^i{}_m = \hat{\nabla}_i Y^i{}_m - \delta_s^\bullet Y^i{}_m \xi^s{}_i - Y^i{}_s \xi^s{}_{mi}. \tag{5.12}$$

By a straightforward calculation, we see that the sum of the last two terms of the right-hand side of (5.12) is zero. Thus (5.9) gives ($K \neq 0$):

$$\nabla_i F_m{}^i = -\mu_1 u_m + \frac{2}{(n+1)} K^{-1} F \delta_m^\bullet H, \tag{5.13}$$

where we have put

$$u^r \nabla_i F_r{}^i = -\mu_1. \tag{5.14}$$

5.2. Deformations

We consider now a deformation of the pseudo-Riemannian structure, i.e. a one-parameter family $t \in [-\varepsilon, \varepsilon]$ of the pseudo-Riemannian metric, leaving fixed the skew-symmetric tensor F_{ij}. The derivative with respect to t of the curvature tensor H of the connection of directions D is defined by (2.42). Thus it follows by contracting i and k, and taking into account (5.8):

$$H'_{jl} = D_i G'^i{}_{jl} - D_l G'^i{}_{ji} - G^i{}_{jrl} G'^r{}_i, \tag{5.15}$$

whence

$$H'(u,u) = H'_{jl} u^j u^l = F^{-2}\left(2D_i G'^i - D_0 G'^i{}_i\right), \tag{5.16}$$

where $D_0 = v^r D_r$. We will prove a lemma analogous to Lemma 3.

Lemma 15. *Let (M, g_t) be a deformation of a pseudo-Riemannian manifold, π_t the connection of directions associated to the pair (ω_t, F) and $\lambda(x)$ a differentiable function on M, then we have the formula*

$$\lambda(x) H'_{ij} u^i u^j = \Psi(F, \lambda) t(u,u) + \text{divergence over } W(M), \tag{5.17}$$

where

$$\begin{aligned}\Psi(F,\lambda) = {} & \tfrac{1}{2}(n+2)F^{-2}\hat{\nabla}_0\hat{\nabla}_0\lambda - \tfrac{3}{2}\nabla^i\nabla_i\lambda + \tfrac{1}{2}(n+1)KF^{-1}F_0^{\ i}\nabla_i\lambda \\ & -(n+2)\lambda\tfrac{1}{4}K^2\left[(n+1)F^{-2}F_{0i}F_0^{\ i} + F_i^{\ j}F_j^{\ i}\right] \\ & +\tfrac{1}{2}(n+2)\lambda K u^r \nabla_j F_r^{\ j}.\end{aligned} \tag{5.18}$$

Proof. By multiplying the two sides of (5.16) by $\lambda(x)$, we will calculate each term of the right-hand side, putting in factor the term $t_{ij}u^iu^j$ we have

$$\lambda F^{-2} D_0 G'^i{}_i = \hat{\nabla}_r\left(\lambda v^r G'^i{}_i F^{-2}\right) - \hat{\nabla}_0\lambda G'^i{}_i F^{-2} - 2\lambda K \xi^s \delta^\bullet_s G'^i{}_i F^{-2}. \tag{5.19}$$

Now the last term of the right-hand side is a divergence. For

$$-2\lambda F^{-2}\xi^s\delta^\bullet_s G'^i{}_i = -2Fg^{sk}\delta^\bullet_s\left(\lambda F^{-3} G'^i{}_i \xi_k\right) = \text{divergence over } W(M). \tag{5.20}$$

On the other hand, by (2.25) we have in the pseudo-Riemannian case

$$G'^i{}_i = \gamma'^i{}_{0i} = \tfrac{1}{2}\hat{\nabla}_0 t^i{}_i. \tag{5.21}$$

Thus

$$\begin{aligned}-\hat{\nabla}_0\lambda G'^i{}_i F^{-2} &= -\tfrac{1}{2}\hat{\nabla}_0\lambda\hat{\nabla}_0 t^i{}_i \\ &= \tfrac{1}{2}F^{-2}\hat{\nabla}_0\hat{\nabla}_0\lambda t^i{}_i + \text{divergence over } W(M).\end{aligned} \tag{5.22}$$

Let us put

$$f = \tfrac{1}{2} F^{-2} \hat{\nabla}_0 \hat{\nabla}_0 \lambda. \tag{5.23}$$

f being differentiable, homogeneous of degree zero in v, we have by (2.18)

$$f \cdot \mathrm{trace}(t) = \left(nf + \tfrac{1}{2} F^2 g^{ij} \delta_i^\bullet \delta_j^\bullet f \right) t(u,u) + \text{divergence over } W(M). \tag{5.24}$$

Taking into account (5.23), the first term of the right-hand side of (5.24) is written as

$$nf + \tfrac{1}{2} F^2 g^{ij} \delta_i^\bullet \delta_j^\bullet f = \tfrac{1}{2} \nabla^i \nabla_i \lambda. \tag{5.25}$$

Thus

$$f \cdot \mathrm{trace}(t) = \tfrac{1}{2} \nabla^i \nabla_i \lambda t(u,u) + \text{divergence over } W(M). \tag{5.26}$$

In virtue of (5.22), (5.26) and (5.20), the formula (5.19) is written as

$$-\lambda F^{-2} D_0 G'^i{}_i = -\tfrac{1}{2} \nabla^i \nabla_i \lambda t(u,u) + \text{divergence over } W(M). \tag{5.27}$$

We now calculate the term

$$2\lambda F^{-2} D_i G'^i = 2\hat{\nabla}_i (\lambda G'^i F^{-2}) - 2\nabla_i \lambda F^{-2} G'^i - 2\lambda \delta_s^\bullet G'^i \delta_i^\bullet \xi^s F^{-2}. \tag{5.28}$$

Now

$$2G'^i = 2\gamma'^i + 2\xi'^i \quad (2\gamma^i = \gamma^i{}_{00}), \tag{5.29}$$

where

$$2\gamma'^i = \hat{\nabla}_0 t^i{}_0 - \tfrac{1}{2} \hat{\nabla}^i t_{00}. \tag{5.30}$$

And F_{ij} supposed independent of t, deriving $2\xi^i$ defined by (5.5), with respect to t, we obtain

$$2\xi'^i = \xi^i t(u,u) - 2t^i{}_m \xi^m, \tag{5.31}$$

where $t = g'$. Thus taking into account (5.29) and (5.30), we have

$$\begin{aligned} &- 2\, \nabla_i \lambda G'^i F^{-2} \\ &\quad = -\nabla_i \lambda \left(\hat{\nabla}_0 t^i{}_0 - \tfrac{1}{2} \hat{\nabla}^i t_{00} \right) F^{-2} + 2t^m{}_i \nabla^i \lambda \xi_m - \xi^i \nabla_i \lambda F^{-2} t(u,u) \\ &\quad = -\hat{\nabla}_0 \left[\nabla_i \lambda t^i{}_0 F^{-2} \right] + \tfrac{1}{2} \hat{\nabla}^i \left[\nabla_i \lambda t(u,u) \right] + 2F g^{mk} \delta_K^\bullet \left[t_{0i} \nabla^i \lambda \xi_m F^{-3} \right] \\ &\qquad - \left[\tfrac{1}{2} \nabla^i \nabla_i \lambda + \xi^i \nabla_i \lambda F^{-2} \right] t(u,u) + \hat{\nabla}_0 \hat{\nabla}_i \lambda t^i{}_0 F^{-2}. \end{aligned} \tag{5.32}$$

The first three terms of the right-hand side are divergences. For the last term we have $t^i{}_0 = g^{ij} t_{j0} = \tfrac{1}{2} g^{ij} \delta_j^\bullet t_{00}$ and obtain easily

$$\begin{aligned} F^{-2} t^i{}_0 \hat{\nabla}_0 \hat{\nabla}_i \lambda = \tfrac{1}{2} &\left[(n+2) \hat{\nabla}_0 \hat{\nabla}_0 \lambda F^{-2} - \nabla^i \nabla_i \lambda \right] t(u,u) \\ &+ \text{divergence over } W(M). \end{aligned} \tag{5.33}$$

Thus (5.32) can be written as

$$-2\nabla_i\lambda G'^i F^{-2} = \left[\tfrac{1}{2}(n+2)\hat{\nabla}_0\hat{\nabla}_0\lambda F^{-2} - \nabla^i\nabla_i\lambda - \xi^i\nabla_i\lambda F^{-2}\right] t(u,u) \\ +\text{divergence over } W(M). \tag{5.34}$$

Let us remark that

$$G'^i{}_s = \gamma'^i{}_{0s} + \xi'^i{}_s. \tag{5.35}$$

We have to calculate now the last term of the right-hand side of (5.28): $-2\gamma G'^i{}_s\xi^s{}_i F^{-2}$. This splits into two terms respectively $-2\lambda\gamma'^i{}_{0s}\xi^s{}_i F^{-2}$ and $-2\lambda\xi'^i{}_s\xi^s{}_i F^{-2}$. The latter can be written, taking into account (5.6) and of $\xi^r{}_i v_r = -\xi_i$:

$$\begin{aligned} &-2\lambda\xi'^i{}_s\xi^s{}_i F^{-2} \\ &= -2\lambda\xi'^i{}_s\xi_{im}g^{ms}F^{-2} = -2F\delta^\bullet_s\left[\lambda\xi'^i\xi_{im}F^{-3}\right]g^{ms} + 6\lambda\xi'^i\xi_i F^{-4} \\ &= (n+2)\lambda\left[t(u,u)\xi^i\xi_i - 2t_{ij}\xi^i\xi^j\right]F^{-4} + \text{divergence over } W(M), \end{aligned} \tag{5.36}$$

where we have put $g^{ms}\xi_{im} = \xi^s{}_i$. On the other hand $t_{ij} = \delta^\bullet_i t_{0j}$ and $\xi_0 = 0$, we have

$$\begin{aligned} &-2(n+2)\lambda t_{ij}\xi^i\xi^j F^{-4} \\ &= -2(n+2)Fg^{ik}\delta^\bullet_i\left[\lambda F^{-5}t_{0j}\xi^j\xi_k\right] + 2(n+2)\lambda t_{0j}\xi^j_i\xi^i F^{-4} \\ &= 2(n+2)\lambda t_{0j}\xi^j{}_i\xi^i F^{-4} + \text{divergence over } W(M). \end{aligned} \tag{5.37}$$

The last term of the right-hand side can be written as

$$\begin{aligned} &2(n+2)\lambda t_{0j}\xi^j{}_i\xi^i F^{-4} \\ &= (n+2)\lambda\delta^\bullet_j t_{00}\xi^j{}_i\xi^i F^{-4} \\ &= (n+2)g^{jk}\delta^\bullet_j\left[\lambda t_{00}\xi_{ik}\xi^i\right]F^{-4} - (n+2)\lambda F^{-4}t_{00}\xi^j{}_i\xi^i{}_j \\ &= \text{divergence} - (n+2)\lambda\left[(n+4)F^{-4}\xi^i\xi_i + F^{-2}\xi^j{}_i\xi^i{}_j\right]t(u,u). \end{aligned} \tag{5.38}$$

In virtue of (5.38) and (5.37) the relation (5.36) is written as

$$-2\lambda\xi'^i{}_s\xi^s{}_i F^{-2} = -\lambda(n+2)\left[(n+3)F^{-4}\xi^i\xi_i + F^{-2}\xi^j{}_i\xi^i{}_j\right]t(u,u) \\ +\text{divergence over } W(M). \tag{5.39}$$

We have now to calculate the expression $-2\lambda\gamma'^i{}_{0s}\xi^s{}_i F^{-2}$:

$$2\gamma'^i = \gamma'^i{}_{00}, \qquad \delta^\bullet_j\gamma'^i = \gamma'^i{}_{0j}.$$

We have

$$\begin{aligned} -2\lambda\gamma'^i{}_{0s}\xi^s{}_i F^{-2} &= -2F^{-2}g^{sm}\delta^\bullet_s(\lambda\gamma'^i\xi_{im}) \\ &= -2Fg^{sm}\delta^\bullet_s(\lambda\gamma'^i\xi_{im}F^{-3}) + 6\lambda\gamma'^i\xi_i F^{-4} \\ &= \delta\hat{Z} + 2(n+2)\lambda\gamma'^i\xi_i F^{-4}, \end{aligned} \tag{5.40}$$

where we have put

$$Z_m = 2\lambda\gamma'^i\xi_{im}F^{-3}, \qquad \hat{Z}_m = Z_m - u_m(Z,u).$$

On the other hand

$$
\begin{aligned}
2(n+2)\lambda\gamma'^{i}\xi_i F^{-4} &= (n+2)\lambda\left[\hat\nabla_0 t^i_0 - \tfrac{1}{2}\hat\nabla^i t_{00}\right]\xi_i F^{-4}\\
&= \delta S - \delta(\sigma u) + \tfrac{1}{2}(n+2)\hat\nabla_i(\lambda\xi^i F^{-2})t(u,u)\\
&\quad -(n+2)F^{-4}\hat\nabla_0(\lambda\xi_i)t'^{i}{}_0,
\end{aligned}
\tag{5.41}
$$

where we have put

$$\sigma = (n+2)\lambda\xi_i t'^{i}{}_0 F^{-3}, \qquad S_i = \tfrac{1}{2}(n+2)\lambda_i t_{00}F^{-4}.$$

Finally, the last term of the right-hand side can be written as

$$
\begin{aligned}
-\tfrac{1}{2}(n+2)F^{-4}\hat\nabla_0(\lambda\xi_i)g^{ij}\delta^\bullet_j t_{00} &= -\tfrac{1}{2}(n+2)Fg^{ij}\delta^\bullet_j\left[\hat\nabla_0(\lambda\xi_i)t_{00}F^{-5}\right]\\
&\quad +\tfrac{1}{2}(n+2)\hat\nabla_j(\lambda\xi^j)F^{-2}t(u,u).
\end{aligned}
$$

Let us put

$$M_i = \tfrac{1}{2}(n+2)\hat\nabla_0(\lambda\xi_i)F^{-3}t(u,u).$$

The relation (5.40) can be written as

$$-2\lambda\gamma'^{i}{}_{0s}\xi^s{}_i F^{-2} = \delta\hat Z + \delta S - \delta(\sigma u) + \delta M + (n+2)\hat\nabla_i(\lambda\xi^i)F^{-2}t(u,u). \tag{5.42}$$

On adding the relation (5.28) and (5.27) and taking into account (5.34), (5.39) and (5.42), we find the expression $\Psi(F,\lambda)$ defined in the lemma. The Ricci tensor H_{jl} defined in Section 5.1 is symmetric, for

$$G^i{}_{ji} = \gamma^i{}_{ji} = \partial_j \mathrm{Log}\sqrt{g}, (g = \det(g_{ij})), \qquad H_{jk} = \tfrac{1}{2}\delta^\bullet_j\delta^\bullet_k H(v,v). \qquad \square$$

In a manner analogous to Lemma 3 of Section 2.5, we have the following lemma.

Lemma 16. *Let (M, g_t) be a deformation of a pseudo-Riemannian manifold, we have*

$$
\begin{aligned}
g^{jk}H'_{jk} &= nH'_{ij}u^i u^j + \tfrac{1}{2}Fg^{jk}\delta^\bullet_j\left[F\delta^\bullet_k\left(H'_{00}/F^2\right)\right]\\
&= nH'_{ij}u^i u^j + \text{divergence over } W(M).
\end{aligned}
$$

The first term of the right-hand side is defined after Lemma 15, on putting $\lambda = 1$.

5.3. Variationals of some scalar curvatures

Let (M, g_t) be a deformation of a compact pseudo-Riemannian manifold and π_t the connection of directions associated to (g_t, F) and H_{jk} corresponding Ricci tensor. Let $\lambda(x)$ be a differentiable function on M. We define on $W(M)$ the action functional as

$$\hat H_t = H_t - \lambda(x)H(u,u), \qquad H_t = g^{ij}H_{ij}. \tag{5.43}$$

We denote as in Section 2.6 by $R^0(g_t)$ the subfamily of the pseudo-Riemannian metrics such that the volume of the corresponding unitary tangent bundle is equal to one for all $t \in [-\varepsilon, \varepsilon]$. We look for a metric $g_t \in R^0(g_t)$ such that the integral $J(g_t)$ is an extremum:

$$J(g_t) = \int_{W(M)} \hat{H}_t \eta_t, \tag{5.44}$$

$$\int_{W(M)} \eta_t = 1. \tag{5.45}$$

By a procedure analogous to one of Section 2.6, using the previous two lemmas, the derivative of $J(g_t)$ with respect to t is defined by

$$J'(g_t) = -\langle C, t \rangle, \tag{5.46}$$

where

$$C_{jk} = H_{jk} - \lambda(x) u_j u_k H(u,u) - \Psi(F, n-\lambda) u_j u_k - H_t \left(g_{jk} - \tfrac{1}{2} n u_j u_k \right), \tag{5.47}$$

and $t = t_{jk}$.

Now the volume of $W(M)$ is constant so that on deriving (5.45) it follows by (2.14) that the tensor D defined by (2.74) is globally orthogonal to t_{ij}. Thus for $g_0 \in R^0(g_t), t = 0$ to give the extremum of the integral, it is necessary and sufficient that there exists a constant b such that we have at $t = 0$:

$$\begin{aligned} &H_{jk} - \lambda(x) u_j u_k H(u,u) - \Psi(F, n-\lambda) u_j u_k - \hat{H} \left(g_{jk} - \tfrac{1}{2} n u_j u_k \right) \\ &\quad = b \left(g_{jk} - \tfrac{1}{2} n u_j u_k \right), \end{aligned} \tag{5.48}$$

where Ψ is defined by (5.18). Multiplying the two sides of (5.48) by v^j and v^k successively we get

$$H(u,u) - \lambda H(u,u) - \Psi(F, n-\lambda) - \hat{H} \left(1 - \tfrac{1}{2} n \right) = b \left(1 - \tfrac{1}{2} n \right). \tag{5.49}$$

On eliminating between (5.48) and (5.49) the expression $\Psi(F, n-\lambda)$ we have

$$H_{jk} - H(u,u) u_j u_k - \hat{H}(g_{jk} - u_j u_k) = b(g_{jk} - u_j u_k). \tag{5.50}$$

Multiplying the two sides by v^j:

$$F^2 H_{0k} = H_{00} v_k. \tag{5.51}$$

On the other hand, since $v^r \delta^\bullet_j H_{rk} = 0$, on deriving with respect to (v^j) the previous relation, we find

$$H_{jk} = H(u,u) g_{jk}. \tag{5.52}$$

Deriving once more vertically (5.52) we have

$$\delta^\bullet_m H_{jk} = \delta^\bullet_m H(u,u) g_{jk},$$

whence, on multiplying the two sides by v^j and v^k:

$$0 = v^j v^k \delta^{\bullet}_m H_{jk} = F^2 \delta^{\bullet}_m H(u,u).$$

Thus, $H(u,u)$ is independent of the direction. (M,g) is therefore a GEM. At the point $t=0$, $H = nH(u,u)$, and $\hat{H}$ reduces to $(n-\lambda)H(u,u)$. On multiplying (5.50) by g^{jk} we then have

$$(1-n+\lambda)H(u,u) = b. \tag{5.53}$$

Eliminating b between (5.53) and (5.49), we get at $t=0$:

$$\left(\tfrac{1}{2}n - \lambda\right) H(u,u) = \Psi(F, n-\lambda). \tag{5.54}$$

We have thus proved the following theorem.

Theorem 6. *The pseudo-Riemannian metric $g_0 \in R^0(g_t)$ at the critical point of the integral $J(g_t)$ defines a GEM.*

For a GEM the scalar curvature H is independent of the direction, (5.13) reduces to

$$\nabla_i F^{ij} = \mu_1 u^j. \tag{5.55}$$

In a particular case, if $\lambda = \frac{1}{2}n$ and $K = 0$, then from (5.53) and (5.54), we have $\Psi = 0$, $C = H(u,u) = b/(1-\frac{1}{2}n)$, $H_{ij} = R_{ij} = Cg_{ij}$. Thus (M,g) is an Einstein manifold.

5.4. Einstein–Maxwell equations

Let M be a differentiable manifold of dimension 4, g the metric tensor of normal hyperbolic type [13], F the closed 2-form of electromagnetism, μ_1 the density of proper electric charge, u the unitary velocity vector time like. The relation (5.55) represents the equation of Maxwell–Lorentz [13]. We have shown that it is derived from the *Bianchi identity* relative to the connection of directions associated to the pair (ω, F) independently from the constant introduced in this connection. On the other hand by (5.7) we obtain the relation between the curvature tensors of π and ω:

$$H^i{}_{jkl} = R^i{}_{jkl} + \hat{\nabla}_k \xi^i{}_{jl} - \hat{\nabla}_l \xi^i{}_{jk} + G^i{}_{jks}\xi^s{}_l - G^i_{jls}\xi^s{}_k + \xi^i{}_{rk}\xi^r{}_{jl} - \xi^i{}_{rl}\xi^r{}_{jk}. \tag{5.56}$$

On contracting i with k and on using the relations (5.6), (5.8) we have

$$H_{jl} = R_{jl} + \hat{\nabla}_i \xi^i{}_{jl} - G^i{}_{jls}\xi^s{}_i - \xi^i{}_{rl}\xi^r{}_{ij}. \tag{5.57}$$

After simplification let it be

$$\begin{aligned} H_{jl} = R_{jl} &+ \tfrac{1}{2}K\left(h_{jl}u^r \nabla_i F_r{}^i + u_l \nabla_i F_j{}^i + u_j \nabla_i F_l{}^i\right) \\ &+ \tfrac{1}{4}K^2\left(g_{jl}F_{ir}F^{ir} + 2F_{jr}F_l{}^r\right). \end{aligned} \tag{5.58}$$

In virtue of the Maxwell–Lorentz equation [13] and on putting $\mu_1 = K_1\rho_1$, where K_1 is constant, we have by changing the indices

$$H_{ij} = R_{ij} - \rho_1 \tfrac{1}{2} K_1 K(g_{ij} + u_i u_j) + \tfrac{1}{4} K^2 \left(g_{ij} F_{rs} F^{rs} + 2F_{ir} F_j{}^r \right). \tag{5.59}$$

Now Maxwell tensor τ_{ij} is of the form [13]

$$\tau_{ij} = \tfrac{1}{4} g_{ij} F_{rs} F^{rs} - F_{ir} F_j{}^r. \tag{5.60}$$

Expressing $2F_{ir}F_j{}^r$ as a function of τ_{ij} and on putting it in (5.59), we have:

$$H_{ij} = R_{ij} - \rho_1 \tfrac{1}{2} K_1 K(g_{ij} + u_i uj) + \tfrac{3}{8} K^2 g_{ij} F_{rs} F^{rs} - \tfrac{1}{2} K^2 \tau_{ij}. \tag{5.61}$$

From (5.59) we obtain

$$g^{ij} H_{ij} = H = R + \tfrac{3}{2} K^2 F_{rs} F^{rs} - \tfrac{5}{2} \rho_1 K_1 K. \tag{5.62}$$

Taking into account the relation, we have

$$H_{ij} - \tfrac{1}{4} H g_{ij} = R_{ij} - \tfrac{1}{2}(R + \lambda) g_{ij} - \chi T_{ij}, \tag{5.63}$$

where we have put

$$T_{ij} = \tau_{ij} + \rho_1 \frac{K_1}{K} u_i u_j, \qquad R + 2\lambda = -\chi \frac{\rho_1 K_1}{K}, \qquad \chi = \frac{K^2}{2},$$

λ and χ are constants. We know that [13] in the case of *pure matter electromagnetic field*, Einstein equation is of the form

$$R_{ij} - \tfrac{1}{2}(R + \lambda) g_{ij} = \chi T_{ij}. \tag{5.64}$$

From (5.63) it follows

$$H_{ij} = \tfrac{1}{4} H g_{ij}. \tag{5.65}$$

Thus (M, g) is a *generalized Einstein manifold*.

Let us now suppose that the equation of Maxwell (5.55) reduces to ($\mu_1 = 0$):

$$\nabla_i F^{ij} = 0. \tag{5.66}$$

Then (5.58) is written in this case as

$$H_{jk} = R_{jk} + \tfrac{1}{4} K^2 \left(g_{jk} F_{ir} F^{ir} + 2F_{jr} F_k^r \right).$$

On making the expression of the tensor τ_{jk} intervene we get

$$H_{jk} - \tfrac{1}{4} H g_{jk} = R_{jk} - \tfrac{1}{4} R g_{jk} - \chi \tau_{jk}.$$

In the case of a *pure electromagnetic field* the right-hand side of this equation vanishes. (M, g) is again a *generalized Einstein manifold*.

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